The certainty principle (review)

D. A. Arbatsky*

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Abstract

The certainty principle (2005) allowed to generalize and unify both the Heisenberg uncertainty principle (1927) and the Mandelshtam-Tamm relation (1945). It turned out to be applicable to any quantum systems, including relativistic quantum fields. One can even suppose, that it will finally turn out to be more fundamental than quantum mechanics and quantum field theory.

In the frame of orthodox quantum theory the certainty principle can be considered as mathematical theorem. Exactly this approach is assumed as a basis in this review. And with it the Heisenberg uncertainty principle and the Mandelshtam-Tamm relation are derived as consequences of the certainty principle. Another interesting consequence appears to be a simple uncertainty relation for angle and angular momentum, which, nevertheless, was not known before.

To wide extent, the certainty principle is not only a mathematical theorem, but also a gnoseological principle. This principle substantially complements the philosophy of quantum mechanics.

Historical comments

Uncertainty principle. A reader interested in the history of the *uncertainty* principle should read, for example, the review [1]. Here I give only some notes, that are important for the following recital.

The uncertainty principle was suggested by Heisenberg in 1927 [2]. Heisenberg formulated it as follows:

- The more precisely the position is determined, the less precisely the momentum is known in this instant, and vice versa.

For uncertainties of coordinate and momentum the following relation was suggested:

$$\Delta_{\rangle} X \, \Delta_{\rangle} P \;\sim\; \hbar \;. \tag{1}$$

Considering concrete examples, Heisenberg gave just qualitative formulation. But for the notion of "uncertainty" he did not give exact definition.

Soon Kennard [3] gave exact mathematical formulation for the case of coordinate and momentum. Assuming "commutation relation" $[X, P] = i\hbar$, he has shown that for an arbitrary quantum state $\rangle$ the following relation takes place:

$$\Delta_{\rangle} X \, \Delta_{\rangle} P \;\geqslant\; \frac{\hbar}{2} \;. \tag{2}$$

And here

$$\Delta_{\rangle} X = \Big\langle \big(X - \langle X \rangle \big)^2 \Big\rangle^{1/2} \;, \qquad \Delta_{\rangle} P = \Big\langle \big(P - \langle P \rangle \big)^2 \Big\rangle^{1/2} \;.$$

It means that, in accordance with probabilistic interpretation of quantum mechanics, it was suggested to understand the uncertainties of coordinate and momentum as standard deviations of these observables.

Because of simplicity of the proof, the relation (2) became conventional mathematical expression of the uncertainty principle and appeared in all textbooks on quantum mechanics. All criticism (including that about correspondence to practical experiment) [1] was mainly ignored. But soon we will see that the certainty principle allows to get

*http://daarb.narod.ru/ , http://wave.front.ru/

alternate inequalities, describing the uncertainty principle, which turn out to be more sapid from practical point of view.

Later the mathematical proof was generalized in many ways. And it was extended to other pairs of non-commuting observables. So, in order to avoid ambiguity in the term *uncertainty principle*, let us give here some other, more up-to-date and general, formulation of this principle[1]:

- If one tries to describe the dynamical state of a *quantum* particle by methods of *classical* mechanics, then precision of such description is limited in principle. The *classical* state of the particle turns out to be *badly* defined. This uncertainty can be mathematically expressed by different inequalities, describing spreads of values of observables that have semiclassical limit.

Of course, Heisenberg could not formulate the uncertainty principle in this form. It would contradict the logic of historical moment. In 1927 physics came from classical mechanics to quantum, and the uncertainty principle was considered as a way from the old theory to new.

But today, when quantum mechanics is already well-established, the uncertainty principle is just a method of qualitative estimation of precision of semiclassical approximation. And there are no reasons to think that this principle is more fundamental than mathematical formalism of quantum mechanics.

Mandelshtam-Tamm relation. So far as for coordinate and momentum there is an uncertainty relation like (1), general arguments, connected with the theory of relativity,[2] point out that, seemingly, an uncertainty relation for time and energy like the following should exist:

$$\Delta_{\rangle} T \, \Delta_{\rangle} H \; \sim \; \hbar \, . \tag{3}$$

Heisenberg himself suggested such a relation. But even in that time it was clear [4] that explanation of such a relation must be more difficult, because there is no such a quantum-mechanical observable as "time", and it is unclear what should be understood as "uncertainty" of time.

Nevertheless, physicists wanted to believe that the uncertainty principle was a *fundamental* physical principle, and attempts to ground relation (3) did not stop up to now. These attempts mainly tried to show that the notion "uncertainty of time" was meaningful. And it was usually made by analysis of measurement process, i. e. to analysis of interaction of quantum particle with apparatus like Heisenberg microscope. (a detailed review with critical analysis see in [5])

I cannot say that those attempts were unsuccessful. Nevertheless, I think that such an approach does not correspond to general methodology of quantum mechanics. The matter is that analytical formalism of quantum mechanics is oriented to studying quantum systems as independent physical objects. This analytical formalism does not study details of interaction of apparatuses with the quantum system.

In 1945 Mandelshtam and Tamm [6] gave rigorous mathematical formulation of relation[3]

$$|\delta t| \, \Delta_{\rangle} H \geqslant \hbar \, . \tag{4}$$

Mandelshtam and Tamm named this relation "the uncertainty relation between energy and time in nonrelativistic quantum mechanics", because, obviously, they thought that they found mathematical expression of the uncertainty principle for energy and time. Correspondingly, further attempts to ground *uncertainty* relation for energy and time were generally attempts to show that the quantity δt in Mandelshtam-Tamm relation can be understood in some sense as "uncertainty" of time.

But, I believe that there are no reasons convincing enough to call Mandelshtam-Tamm relation as *uncertainty* relation. As it was explained above, the uncertainty principle is not more fundamental than mathematical formalism of quantum mechanics itself. And what is more, analysis of relativistic quantum systems shows that this principle is *less* fundamental. Therefore there are no reasons to think that fundamental uncertainty relation like (3) must exist.

Soon we will see that Mandelshtam-Tamm relation in the case of closed systems is a consequence of other physical principle, the *certainty* principle, which is more fundamental than the Heisenberg uncertainty principle. And the

[1] Such a formulation not only allows to better understand, what exactly is "uncertain", but also allows to clearly see the contrast with the *certainty* principle.

[2] But soon we will see that these arguments are *incorrect*, because existence of the uncertainty principle is connected with specific peculiarities of *non-relativistic* approximation.

[3] Here it is written a little differently, so that connection with further discussion would be clearer.

quantity δt in the inequality (4), if we make transformation from Schrödinger to Heisenberg representation (or to RCQ-representation), is a logarithmic coordinate on Poincare group. For some purposes this coordinate can be interpreted as "uncertainty of time", but, generally speaking, there is no need in such interpretation.

As regards non-closed systems, with arbitrary dependence of Hamiltonian on time, Mandelshtam-Tamm relation should be considered as relation for the speed of quantum evolution. And it is quite independent relation, not mathematically equivalent to the certainty relation for time and energy.

Certainty principle. When we raise a question about application of ideas connected with the theory of relativity in quantum mechanics, it is worth to think about what we know about relativistic invariance of quantum theory at all.

The group of invariance of Minkowski space[4] is Poincare group. With invariance with respect to this group of laws of nature, in general, and of quantum mechanics, in particular, is connected the notion of relativistic invariance.

All known physically important quantum systems, to which action of Poincare group can be applied, are quantum fields (or their subsystems). And before 2002 mathematical construction of quantum fields was made by such recipes[5], which made action of Poincare group on space of states of quantum field hidden. In fact, physicists were satisfied by statement that Poincare group acts on space of states not-explicitly, by some quite complicated formulas.

When relativistic canonical quantization (RCQ) appeared [8] (for introduction review [9] is recommended) it became clear how action of Poincare group is transferred from Minkowski space to the space of states of quantum field.[6] It became also obvious the following:

- The notion of coordinate as quantum-mechanical observable for relativistic systems is not natural at all, not even non-fundamental. (Attempts in the past to introduce coordinates for some systems, for example [10], seem to be quite artificial.)
- For this reason the uncertainty principle *can not* be fundamental, in the sense of its generalization for relativistic quantum theory.

Using methodology of RCQ, I have formulated [12] the *certainty* principle:

- If one describes the dynamical state of a quantum particle (system) by methods of *quantum* mechanics, then the *quantum* state of the particle (system) turns out to be *well* defined. This certainty of the quantum dynamical state means that "small" space-time transformations can not substantially change the quantum state. And for the case of Poincare group, for transformations, that can substantially change the quantum state, we have estimation[7]:
$$\Delta_{\rangle} \left(-\delta x_\mu \, P_\mu + \tfrac{1}{2} \, \delta\omega_{\mu\nu} \, J_{\mu\nu} \right) \geqslant \hbar \, , \tag{5}$$
Here P_μ is a vector operator of energy-momentum, $J_{\mu\nu}$ is a tensor operator of the four-dimensional angular momentum, δx_μ and $\delta\omega_{\mu\nu}$ are the standard logarithmic coordinates of the Poincare group.

So far as the certainty principle is universal, it works well also in usual quantum mechanics, including *non-relativistic*. So, it seems to be surprising that it was not formulated earlier. Nevertheless, there are two explanations for this.

First, this principle is formulated naturally with basic notions of RCQ. In particular, it is most natural in formulation of the certainty principle to understand the notion of *dynamical state* within RCQ interpretation. This difficulty, seemingly, was the main.

Second, in order to come to the certainty principle from the usual quantum mechanics, it is necessary to use the theory of quantum angle (Fubini-Study metric). But existence of this metric, though not difficult to prove, is not obvious. This metric was discovered only in 1905 by Fubini [13] and Study [14], and most physicists do not know about it.

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[4] Of four-dimensional space-time of the special theory of relativity.

[5] Recipes, because formal mathematical process did not exist.

[6] Specific feature of RCQ method is construction of quantum fields in the frame of four-dimensional geometry of Minkowski space, without artificial separation of space and time.

[7] Here and later we imply relativistic summation by pairs of the same Greek indices:
$$a_\mu \, b_\mu = a_0 \, b_0 - a_1 \, b_1 - a_2 \, b_2 - a_3 \, b_3 \, .$$

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Mathematical formalism

Quantum angle (Fubini-Study metric). Consider the unit sphere $\|x\| = 1$ in the usual three-dimensional Euclidean space $\mathbb{R}^3$. The distance measured on the surface of the sphere between any two points is equal to the angle between rays starting from the center of coordinates and coming through these points.

So, the connection of angles with rays on the sphere makes obvious the important property of angle: it satisfies the "triangle inequality":

$$\angle(a,c) \leqslant \angle(a,b) + \angle(b,c) \ . \tag{6}$$

It is known, that angle in this example is expressed through the scalar product $\langle\cdot|\cdot\rangle$ in the space $\mathbb{R}^3$ by formula:

$$\angle(a,b) \ = \ \arccos \langle a|b\rangle \ .$$

Here in the right part of the formula vectors, whose ends are in the points of intersections of the rays a and b with the unit sphere, are denoted by the same letters as rays. Possible values of angle in this example are in the range from 0 to π .

Let us now consider angles not between rays, but between lines coming through the center of coordinates. In this case possible values of angle are in the more narrow range from 0 to $\pi/2$. The formula for the angle in this case is slightly changed:

$$\angle(a,b) \ = \ \arccos \bigl|\langle a|b\rangle\bigr| \ . \tag{7}$$

Will the angle satisfy the triangle inequality (6) in this case? The answer to this question does not seem to be immediately obvious, because for every line we have two points on the sphere. But elementary logical reasoning allows to reduce this example to the previous. The answer turns out to be positive.

Consider now the case of the three-dimensional complex Hilbert space $\mathbb{C}^3$. Instead of real lines in this case we obviously have complex lines, i. e. one-dimensional complex subspaces. Every such a subspace intersects with the unit sphere even not in two, but in infinite number of points. These points are different from each other by complex factor, whose absolute value is equal to one.

The angle in this case is defined by the same formula (7), but the scalar product here corresponds to the space $\mathbb{C}^3$. Possible values of angle, obviously, also are in the range from 0 to $\pi/2$.

Will angle in this case satisfy the triangle inequality (6)? The answer in this case turns out to be positive also. This fact was proved by Fubini [13] and Study [14] (quite simple proof is also given in [12]), so the angle between complex lines in Hilbert space is called *Fubini-Study metric*.

Of course, the fact that triangle inequality takes place cannot depend on the dimensionality of the Hilbert space, because when we check the triangle inequality we always can consider the corresponding three-dimensional subspace.

So far as in quantum mechanics the space of states of a quantum system $\mathcal{H}$ is complex Hilbert space, Fubini-Study metric can be naturally considered as a measure of difference of quantum states. In this context we will call this metric *quantum angle*, because it allows to form natural collocations like "quantum angular speed".

Quantum angular speed. Let now the vector r depend on the real parameter t : $t \in \mathbb{R}$, $r(t) \in \mathcal{H}$, $\| r(t) \| = 1$.

Let us define *the quantum velocity* $v(t)$ by the formula:

$$v(t) = \dot{r}(t) = \lim_{\delta t \to 0} \frac{r(t+\delta t) - r(t)}{\delta t} \ .$$

Let us define also *the quantum angular speed* $\omega(t)$ by the formula:

$$\omega(t) = \lim_{\delta t \to 0} \frac{\angle\bigl(r(t+\delta t) \, , \, r(t) \bigr)}{|\, \delta t \,|} \ .$$

In order to express $\omega(t)$ through $v(t)$ let us decompose $v(t)$ into the two orthogonal components:

$$v_{\|}(t) = r(t)\,\langle r(t)|v(t)\rangle \ , \qquad v_{\perp}(t) = v(t) - v_{\|}(t) \ .$$

This decomposition formally looks the same as the corresponding decomposition in the real case, which is well known in kinematics of point. But we should not rely on this analogy too much. The matter is that in the real case existence of the parallel component of velocity is necessarily connected with growth of the norm of the vector. But in the complex case under consideration vector is supposed to have always unit norm. Existence of parallel component of velocity turns out connected with the possibility of multiplication of the vector by complex phase factor, that is in absolute value equal to one. Nevertheless, the following theorem is true

T h e o r e m. *The quantum angular speed is equal to the norm of the orthogonal component of the quantum velocity:*

$$\omega(t) = \| \, v_\perp(t) \, \| \; . \tag{8}$$

Before proving this theorem let us note that in the real case equality (8) is *definition* of angular speed ω . And angle in the real case is *defined* as integral of angular speed along extreme arc. After that trigonometrical function arccos is *defined* for *finite* angles.

Straightforward transfer of such logic to the case of complex Hilbert space is problematic, because there are always many extreme arcs, and none of them is so simple that appearance of arccos function would be geometrically obvious.

So, here we need to study directly infinitesimal angles, i. e. work around singular point of arccos function. In order to escape from this, let us use Parseval equality to change arccos to arcsin :

$$\angle\big(\, r(t+\delta t) \, , \, r(t) \,\big) \;=\; \arccos \big| \, \langle \, r(t+\delta t) \, | \, r(t) \, \rangle \, \big| \;=$$

$$=\; \arcsin \big\| \, r(t+\delta t) - r(t) \, \langle \, r(t) \, | \, r(t+\delta t) \, \rangle \, \big\| \;=$$

$$=\; \arcsin \big\| \, r(t+\delta t) \;-\; r(t) \;-\; r(t) \, \langle \, r(t) \, | \, r(t+\delta t) - r(t) \, \rangle \, \big\| \;=$$

$$=\; \arcsin \big\| \, v(t) \, \delta t \;+\; o(\delta t) \;-\; r(t) \, \langle \, r(t) \, | \, v(t) \, \delta t + o(\delta t) \, \rangle \, \big\| \;=$$

$$=\; \arcsin \Big\| \, \big(v(t) \;-\; r(t) \, \langle \, r(t) \, | \, v(t) \, \rangle \big) \, \delta t \;+\; o(\delta t) \, \Big\| \;=$$

$$=\; \arcsin \big\| \, v_\perp(t) \, \delta t \;+\; o(\delta t) \, \big\| \;=\; \arcsin \big(\, \|v_\perp(t)\| \, |\delta t| \;+\; o(\delta t) \, \big) \;=$$

$$=\; \|v_\perp(t)\| \, |\delta t| \;+\; o(\delta t) \; .$$

Dividing by $|\delta t|$, we get (8). ■

Continuous unitary groups. Suppose now that we have in the space of states of quantum system a self-adjoint operator $A = A^*$. The set of unitary operators of type $U(\delta s) = e^{-\, i \, \delta s \, A / \hbar}$ (where δs runs the set of all real numbers, and $\hbar$ is fixed positive real number (in quantum mechanics this is Planck's constant)) forms strongly continuous one-parameter unitary group: $U(\delta s_1) \, U(\delta s_2) \;=\; U(\delta s_1 {+} \delta s_2)$. The operator A here is called *generator* of this group.

And suppose that the state vector is changed by the action of this group:

$$r(\delta s) \;=\; | \, \delta s \, \rangle \;=\; U(\delta s) \, \rangle \;=\; e^{-\, i \, \delta s \, A / \hbar} \, \rangle \; .$$

Here $| \, \delta s \, \rangle \in \mathcal{H}$ is another notation of the vector with parameter equal to δs ; $\rangle \in \mathcal{H}$ is a fixed ket-vector.

Suppose now that the function $r(\delta s)$ is differentiable. Then the quantum velocity is expressed by the formula:

$$v(\delta s) \;=\; \tfrac{1}{i \hbar} \, A \, e^{-\, i \, \delta s \, A / \hbar} \, \rangle \;=\; \tfrac{1}{i \hbar} \, A \, | \, \delta s \, \rangle \; .$$

The mean of the operator A does not depend on time:

$$\overline{A} \;=\; \langle \, \delta s \, | \, A \, | \, \delta s \, \rangle \;=\; \langle \, e^{+\, i \, \delta s \, A / \hbar} \, A \, e^{-\, i \, \delta s \, A / \hbar} \, \rangle \;=\; \langle \, A \, e^{+\, i \, \delta s \, A / \hbar} \, e^{-\, i \, \delta s \, A / \hbar} \, \rangle \;=\; \langle \, A \, \rangle \; .$$

Therefore the components of the quantum velocity can be written just as:

$$v_{\|}(\delta s) \;=\; | \, \delta s \, \rangle \, \langle \, \delta s \, | \, \tfrac{1}{i \hbar} \, A \, | \, \delta s \, \rangle \;=\; \tfrac{1}{i \hbar} \, \overline{A} \, | \, \delta s \, \rangle$$

$$v_\perp(\delta s) \;=\; \tfrac{1}{i \hbar} \, \big(A - \overline{A} \big) \, | \, \delta s \, \rangle \; .$$

The quantum angular speed turns out to be independent of the parameter also:

$$\omega(\delta s) \;=\; \| \, v_\perp(\delta s) \, \| \;=\; \tfrac{1}{\hbar} \, \langle \, \delta s \, | \, \big(A - \overline{A} \big)^2 \, | \, \delta s \, \rangle^{1/2} \;=\; \tfrac{1}{\hbar} \, \langle \, e^{+\, i \, \delta s \, A / \hbar} \, \big(A - \overline{A} \big)^2 \, e^{-\, i \, \delta s \, A / \hbar} \, \rangle^{1/2} \;=$$

$$= \tfrac{1}{\hbar} \langle \left(A - \overline{A}\right)^2 e^{+i\,\delta s\,A/\hbar}\, e^{-i\,\delta s\,A/\hbar} \rangle^{1/2} = \tfrac{1}{\hbar} \langle \left(A - \overline{A}\right)^2 \rangle^{1/2} = \tfrac{1}{\hbar}\, \Delta_{\rangle} A\ .$$

Here $\Delta_{\rangle} A$ is a short notation for the standard deviation of A in the state $\rangle$.

So, we have proved the important

T h e o r e m. *Standard deviation of the generator of one-parameter unitary group is equal to the quantum angular speed, multiplied by Planck's constant* $\hbar$. *And both quantities remain conserved under the action of the group.*

Determine now, how different can become the initial $\rangle$ and final $|\,\delta s\,\rangle$ vectors. The triangle inequality, in usual way, is generalized to arbitrary curves [12]. Therefore, the angle between initial and final vectors cannot be greater than increment of parameter, multiplied by quantum angular speed:

$$\angle\left(\,|\,\delta s\,\rangle\,,\,\rangle\,\right) \leqslant |\delta s|\,\tfrac{1}{\hbar}\,\Delta_{\rangle} A\ . \tag{9}$$

For physical applications it is convenient to introduce some qualitative border, when two state vectors can be considered different *substantially.* So, we will say that two vectors are different *substantially*, if the angle between them is greater or equal to 1 .

Let us reformulate inequality (9) as the following theorem.

T h e o r e m. (certainty principle) *So that under the action of strongly continuous one-parameter unitary group* $U(\delta s) = e^{-\,i\,\delta s\,A/\hbar}$ *the initial state vector* $\rangle$ *changes substantially, it is necessary to satisfy the inequality:*

$$|\delta s|\,\Delta_{\rangle} A \geqslant \hbar\ . \tag{10}$$

This theorem is easily generalized to the case when the group is many-parameter. Namely, if δs and A have matrix indices, implying summation, then the inequality (10) can be re-written for this case a little differently:

$$\Delta_{\rangle}(\,\delta s_j\,A_j\,) \geqslant \hbar\ .$$

And here δs_j are the so called logarithmic coordinates of the group.

It is worth also to mention the following. When proving the theorem we supposed that quantum velocity is well-defined, i. e. the corresponding derivative of the vector with respect to the parameter exists and belongs to the Hilbert space. But, if the generator is an unbounded operator, for some state vectors it can be not so. Nevertheless, the statement of the theorem turns out to be true even in this case, because in this case the value $\Delta_{\rangle} A$ turns out to be infinite. Therefore, when parameter is not equal to zero, in the left part of the inequality we have infinity (which is, of course, greater than Planck's constant).

So, the statement of the theorem turns out to be true for *any* state vector $\rangle$ and *any* (self-adjoint) generator A .

Subgroups of the Poincare group

The theorem presented at the end of the previous section is the most general formulation of the certainty principle. But in different situations the unitary group under consideration can be representation of some concrete physical symmetry group. The most important example is Poincare group, for which the general certainty relation (5) was already presented above. This group has some physically important subgroups, which are so practically important, that we will discuss them here in more detail.

Space translations. It is known that generators of three parameter group of space translations are components of the vector operator of momentum P_1 , P_2 and P_3 . Consequently, the certainty relation is written as

$$\Delta_{\rangle}(\,\delta x_1\,P_1 + \delta x_2\,P_2 + \delta x_3\,P_3\,) \geqslant \hbar\ . \tag{11}$$

Here numbers δx_1 , δx_2 and δx_3 form components of the vector that defines the translation.

If we suppose now that $\delta x_2 = \delta x_3 = 0$, and also omit the index, then the inequality (11) can be written as:

$$|\delta x|\,\Delta_{\rangle} P \geqslant \hbar\ .$$

Suppose now, that for the system under consideration we were able to find some observable X , that can be considered in some sense a “coordinate operator”. This supposition is accepted as undoubted in non-relativistic quantum mechanics. In contrast, in relativistic quantum mechanics there is no natural notion of coordinate.[8] It is possible, for example, to form from generators of the Poincare group some vector operator, to which in classical mechanics coordinates of the center of mass correspond. But it should be understood that it will have some unusual properties. For example, its coordinates will not commute with each other.

Suppose that X is a self-adjoint operator with continuous spectrum, $X = X^*$. Let us denote $\Omega_{(a,b)}$ its spectral projector[9] for an arbitrary real interval (a,b) .

Suppose also that X , being a coordinate, behaves in usual way under action of translations, namely, for any a , b and δx we have equality:

$$e^{+\,i\,\delta x\,P/\hbar}\,\Omega_{(a+\delta x,b+\delta x)}\,e^{-\,i\,\delta x\,P/\hbar} \;=\; \Omega_{(a,b)}\ ,$$

i. e. P is a "generator of spectral shifts" for X .

Suppose now that the system is in state $\rangle$, $\langle | \rangle = 1$.

The quantity $\langle\, \Omega_{(a,b)}\, \rangle$, obviously, defines the probability to find the system inside the interval (a,b) . Let us define such l and r , that

$$\langle\, \Omega_{(-\infty,l)}\, \rangle \;=\; \langle\, \Omega_{(r,+\infty)}\, \rangle \;=\; \tfrac{1-\sin 1}{2} \;\approx\; 0,07926\ldots$$

It is easy to see, that l and r exist[10]. So, the quantity $\delta_\rangle X = r - l$ can be naturally called "uncertainty" of the coordinate X .

T h e o r e m. (uncertainty principle) *The following inequality takes place:*

$$\delta_\rangle X \Delta_\rangle P \geqslant \hbar\ . \tag{12}$$

In order to prove this theorem let us use (two times) the triangle inequality for the quantum angle:

$$\begin{aligned}\angle\big(\,\rangle\ ,\ e^{-\,i\,\delta_\rangle X\,P/\hbar}\rangle\,\big) \;\geqslant\; &\angle\big(\,\Omega_{(-\infty,r)}\rangle\ ,\ \Omega_{(r,+\infty)}\,e^{-\,i\,\delta_\rangle X\,P/\hbar}\rangle\,\big) \;-\\ &-\angle\big(\,\rangle\ ,\ \Omega_{(-\infty,r)}\rangle\,\big) \;-\; \angle\big(\,e^{-\,i\,\delta_\rangle X\,P/\hbar}\rangle\ ,\ \Omega_{(r,+\infty)}\,e^{-\,i\,\delta_\rangle X\,P/\hbar}\rangle\,\big) \;=\\ =\; \tfrac{\pi}{2} \;-\; \arcsin\sqrt{\tfrac{1-\sin 1}{2}} \;-\; \arcsin\sqrt{\tfrac{1-\sin 1}{2}} \;&=\; \tfrac{\pi}{2} \;-\; \big(\tfrac{\pi}{4}-\tfrac{1}{2}\big) \;-\; \big(\tfrac{\pi}{4}-\tfrac{1}{2}\big) \;=\; 1\ .\end{aligned}$$

But this means that under action of $e^{-\,i\,\delta_\rangle X\,P/\hbar}$ the vector $\rangle$ changes substantially.

Applying the *certainty* principle, we directly get (12). ■

So, the certainty principle allows to get the alternate inequality (12) describing the uncertainty principle. This inequality is not equivalent to the Kennard inequality (2).

From formal mathematical point of view, none of the inequalities (2) and (12) is stronger than the other.

But it is not difficult to see that the fraction of the quantities $\Delta_\rangle X$ and $\delta_\rangle X$ always satisfies the inequality:

$$\frac{2\,\Delta_\rangle X}{\delta_\rangle X} \;\geqslant\; \sqrt{1-\sin 1} \;\approx\; 0,398\ldots$$

Therefore, if we weaken the Kennard inequality (2) by the additional factor $0,398$ (which is usually not very important for qualitative estimations), then it becomes weaker than (12).

On the other hand, the fraction $\Delta_\rangle X/\delta_\rangle X$ can be arbitrarily great, or even infinitely great, if the wave packet is badly localized. In such a situation the Kennard inequality (2), in contrast to the inequality (12), does not give any estimation for the uncertainty of momentum at all.

So, the inequality (12) is more informative than (2) for qualitative estimations.

[8] This is true even for Dirac's electron. Superposition of positive- and negative-frequency solutions of Dirac's equation does not have any physical meaning, and therefore there is no observable described by "operator of multiplication by variable x ".

[9] Roughly speaking, spectral projector is an operator nullifying wave function in X -representation outside of the given interval.

[10] But, generally speaking, they are not unique. In order to eliminate this non-uniqueness, it is convenient to choose l maximum of the possible, and r — minimum. Then the distance $r-l$ will be minimum.

Space rotations. The generators of the three-parameter group of space rotations are components of the vector operator of angular momentum J_1 , J_2 and J_3 . Consequently, the certainty relation is written as

$$\Delta_\rangle(\,\delta\varphi_1\,J_1 \,+\, \delta\varphi_2\,J_2 \,+\, \delta\varphi_3\,J_3\,) \geqslant \hbar\ .$$

Logarithmic coordinates have here simple geometric interpretation: rotation, described by coordinates $\delta\varphi_1$, $\delta\varphi_2$ and $\delta\varphi_3$, can be performed as rotation around the vector with these coordinates by angle equal to the length of this vector.

If we suppose now that $\delta\varphi_2 = \delta\varphi_3 = 0$ and omit the index, then we have:

$$|\delta\varphi|\,\Delta_\rangle J \geqslant \hbar\ .$$

Let us try to get an uncertainty relation by the same scheme as in the case of the group of space translations.

First of all, it is necessary to note, that *uncertainty* relation in the form "product of uncertainty of angle and uncertainty of angular momentum is greater or equal to Planck's constant" can not be suggested, because in a state, when uncertainty of angular momentum is equal to zero (an eigenstate), "uncertainty of angle" should not be greater than 2π .

Furthermore, here we meet the difficulty that it is impossible to introduce a "good" self-adjoint operator of angle. The number of operators turns out to be infinite (if at least one is defined), but as soon as we fix the choice of one of them by some condition (for example, that its spectrum is inside the interval from 0 to 2π), it turns out to have quite complicated behaviour under rotations, and analysis of formulas becomes difficult.

So, we will not choose here one from the infinite set of operators of angle, but will imply that we talk about the whole infinite set. For this infinite set we will use symbolic notation Φ .

Such an approach does not meet difficulties, because when deriving uncertainty relation we in fact operate with spectral measure (the set of spectral projectors). And this spectral measure for all operators of angle is the same in essence. Just argument of this measure is not an interval of the real line, as in the case of coordinate operator, but a segment of circle. And real coordinate on the circle should be understand up to addition of 2π . So, the spectral measure satisfies relation:

$$\Omega_{(a+2\pi,b+2\pi)} = \Omega_{(a,b)}\ .$$

Suppose also that the operator J of angular momentum is a "generator of spectral rotations" for Φ :

$$e^{+\,i\,\delta\varphi\,J/\hbar}\,\Omega_{(a+\delta\varphi,b+\delta\varphi)}\,e^{-\,i\,\delta\varphi\,J/\hbar} \;=\; \Omega_{(a,b)}\ .$$

The quantity $\langle\,\Omega_{(a,b)}\,\rangle$, obviously, defines the probability to find the system inside the angle segment (a,b) . Suppose now that there is such a small segment (l,r) , where mainly the probability to find the system is concentrated. Namely, suppose that the probability to find the system in the alternate segment is:

$$\langle\,\Omega_{(r,l+2\pi)}\,\rangle \;=\; \tfrac{1-\sin 1}{2} \;\approx\; 0,07926\ldots$$

If r and l were chosen so that the quantity $\delta_\rangle\Phi \;=\; r-l$ is minimum, then $\delta_\rangle\Phi$ can be naturally called "uncertainty" of the angle Φ .

T h e o r e m. (uncertainty principle for angle and angular momentum) *The following inequality takes place:*

$$\delta_\rangle\Phi \;\geqslant\; \min\;(\;\hbar\,/\,\Delta_\rangle J\;;\;\pi\;)\ . \tag{13}$$

Consider the case, when $\delta_\rangle\Phi \;\leqslant\; \pi$. In this case the rotated angle segment (l,r) will not overlap with the non-rotated one, and we can apply the same estimation technique for quantum angle as in the case of translations.

So, let us use (two times) the triangle inequality for the quantum angle:

$$\angle\,(\,\rangle\ ,\ e^{-\,i\,\delta_\rangle\Phi\,J/\hbar}\,\rangle\,)\;\geqslant\;\angle\,(\,\Omega_{(l,r)}\,\rangle\ ,\ \Omega_{(r,r+\delta_\rangle\Phi)}\,e^{-\,i\,\delta_\rangle\Phi\,J/\hbar}\,\rangle\,)\,-$$

$$-\,\angle\,(\,\rangle\ ,\ \Omega_{(l,r)}\,\rangle\,)\;-\;\angle\,(\,e^{-\,i\,\delta_\rangle\Phi\,J/\hbar}\,\rangle\ ,\ \Omega_{(r,r+\delta_\rangle\Phi)}\,e^{-\,i\,\delta_\rangle\Phi\,J/\hbar}\,\rangle\,)\;=$$

$$=\;\tfrac{\pi}{2}\;-\;\arcsin\sqrt{\tfrac{1-\sin 1}{2}}\;-\;\arcsin\sqrt{\tfrac{1-\sin 1}{2}}\;=\;\tfrac{\pi}{2}\;-\;\left(\tfrac{\pi}{4}-\tfrac{1}{2}\right)\;-\;\left(\tfrac{\pi}{4}-\tfrac{1}{2}\right)\;=\;1\ .$$

But this means that under action of $e^{-\,i\,\delta_\rangle\Phi\,J/\hbar}$ the vector $\rangle$ is changed substantially.

Applying the *certainty* principle, we directly get (13). ■

We see that the certainty principle allows to easily get also the simple *uncertainty* relation for angle and angular momentum.

It should be noted that attempt [16] to get an uncertainty relation for angle and angular momentum by analogy with the Kennard inequality (2) leads to quite complicated formulas. Namely, if we fix the self-adjoint operator of angle Φ by the condition, that its spectrum is the interval $[-\pi, \pi]$, and define the "uncertainty" of angle by the formula:

$$\Delta'_{\rangle}\Phi = \min_{\delta\varphi} \sqrt{\langle\, e^{+\,i\,\delta\varphi\, J/\hbar}\, \Phi^2\, e^{-\,i\,\delta\varphi\, J/\hbar}\,\rangle} \ ,$$

then we have the uncertainty relation:

$$\Delta'_{\rangle}\Phi\, \Delta_{\rangle} J \geqslant \frac{\hbar}{2} \left[1 - \frac{3}{\pi^2}\, (\Delta'_{\rangle}\Phi)^2 \right] \ . \tag{14}$$

For investigation of semiclassical limit the inequality (13) turns out to be more convenient than (14), because allows to use weaker conditions for localization of wave packet.

Time shifts. Poincare group includes as a subgroup one-parameter group of shifts in time. It is natural to expect that the certainty principle has corresponding mathematical expression in this case. And this is really so.

But our discussion here is complicated by the circumstance that all educational literature on both classical and quantum mechanics is oriented to resolving of problems of dynamics only. And the notion of dynamical state is necessarily attributed to some moment of time. Such approach hides relativistic invariance. So, the notion of time shifts becomes senseless.

Completely self-consistent, from this point of view, are invariant Hamiltonian formalism (in the case of the classical mechanics) and relativistic canonical quantization (RCQ) (in the case of relativistic quantum field theory). In these approaches the theory includes not only time shifts, but also Lorentz boosts. But, so far as most physicists are not well-acquainted with such approaches now, here we will discuss some intermediate approach that is in essence based on Heisenberg representation. (Lorentz boosts we will not consider in this review at all.)

Usually, dynamical state S_t in some moment of time t is understood as some set of values, that can be measured in the moment t by some set of apparatuses, and that define the whole subsequent evolution of the system (i. e. analogous sets of values in subsequent moments of time). In classical mechanics dynamical state is usually described either by set of coordinates and velocities (Lagrangian formalism) or by set of coordinates and momenta (Hamiltonian formalism). In other words, dynamical state in a given moment of time is defined by a point on some manifold, which is called phase space. In quantum mechanics dynamical state is described by vector in Hilbert space.

When we have the question about description of the group of space motions on the space of dynamical states in some moment of time, we usually think in the following way. Suppose a set of physical apparatuses K , which measures the full set of dynamical variables of the system under consideration, under action of some motion G transferred to K' , $G : K \to K'$. And suppose for some dynamical state of the system S we have dynamical state S' , such that the set of apparatuses K' , which measures parameters of the state S' , gives the same results as K , which measures parameters of the state S . Then we say that under action of the transformation G state S transfers to S' . Symbolically we can write it as:

$$G : K \to K' \ , \ K(S) = K'(S') \qquad \Longrightarrow \qquad G : S \to S' \ .$$

Exactly in this way the action of space translations and rotations is introduced, described above.

It seems, it would be natural to apply this definition to the group of time shifts:

$$G : K_{t_1} \to K_{t_2} \ , \ K_{t_1}(S_{t_1}) = K_{t_2}(S_{t_2}) \qquad \Longrightarrow \qquad G : S_{t_1} \to S_{t_2} \ .$$

But this definition turns out to be not very useful, because it turns out to be purely formal and with no connection with dynamics.

Let us improve a little the definition of dynamical state given above. Let us suppose that if some dynamical state S_{t_1} is transferred by evolution to S_{t_2} , then S_{t_1} and S_{t_2} describe the same dynamical state. In other words, dynamical state is just some abstract set of values (not necessarily connected with measurements in a concrete moment of time) defining evolution of the system. When we talk about quantum systems, we can think that dynamical state is just described by Heisenberg vector of state.

If the system is closed, its Hamiltonian does not depend on time, and it commutes with space motions. Therefore the theory of space motions, described in the previous sections, can be applied in this case. And what is more, the group of time shifts is joined here, which is described by the family of operators:

$$U(\delta t) = e^{-\,i\,\delta t\,(-H)/\hbar} \ .$$

It should be noticed that generator of this group is Hamiltonian with sign "minus". An operator, describing action of the group under consideration, is reciprocal of the operator, describing evolution in Schrödinger equation. We can say that here the group acts actively, but in the Schrödinger representation it acts passively.

The certainty relation for the group of time shifts takes the form:

$$|\delta t|\,\Delta_{\rangle} H \geqslant \hbar \ . \tag{15}$$

If we look at this relation from the point of view of the Schrödinger representation (but doing so we lose connection with Poincare group), then this relation turns out to be exactly the Mandelshtam-Tamm relation [6] for the speed of quantum evolution.

So, the Mandelshtam-Tamm relation for the case of closed systems is a consequence of the certainty principle.

Let us consider now non-closed systems. The Mandelshtam-Tamm relation for the speed of quantum evolution (derived in the Schrödinger representation) is naturally generalized for this case. Let us denote Hamiltonian for this case as H^{full}. So far as the quantity $\Delta_{|t\rangle} H^{\text{full}}$, generally speaking, depends on time, it should be just averaged:

$$|\tau|\,\overline{\Delta_{|t\rangle} H^{\text{full}}} \geqslant \hbar \ . \tag{16}$$

Here τ is the time interval of evolution (it does not have meaning of logarithmic coordinate on any group).

The certainty principle in this situation, formally, turns out to be not applicable. But, in practical situations, when we consider non-closed quantum systems, their Hamiltonian can be usually represented as sum of two terms, the Hamiltonian of "free" system and Hamiltonian of interaction "with external classical field":

$$H^{\text{full}} = H + H^{\text{int}} \ .$$

And usually the Hamiltonian of interaction H^{int} can be "turned off".

In this case it is natural to connect the notion of dynamical state of the system with its evolution when Hamiltonian of interaction is turned off (though, the state can be considered in a concrete moment of time, without turning off the Hamiltonian of interaction). Correspondingly, the certainty relation (15) remains the same. Its meaning becomes somewhat more abstract, with no direct connection with the Mandelshtam-Tamm relation (16).

In the end, let us note that Mandelshtam and Tamm derived their relation by direct estimation of the scalar product of the initial and final vectors of state. They did not use the theory of Fubini-Study metric. Such a derivation turns out to be substantially shorter. But geometrical nature of the result turns out to be quite hidden. This additionally complicates interpretation (in addition to the more serious disadvantage: interpretation based on the Schrödinger equation).

Derivation of the Mandelshtam-Tamm relation (in the frame of the Schrödinger equation), based on Fubini-Study metric, was given later by Anandan and Aharonov [18]. They did not give the reference to Mandelshtam and Tamm. So, in the western literature this relation is often erroneously called the Anandan-Aharonov relation.

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	uncertainty principle (Heisenberg, 1927)	$\Longleftarrow$ **certainty principle** (Arbatsky, 2005)
$x,\ p$	$\Delta_\rangle X\, \Delta_\rangle P \geqslant \frac{\hbar}{2}$ (Kennard, 1927) $\delta_\rangle X\, \Delta_\rangle P \geqslant \hbar$ (Arbatsky, 2006)	$\lvert\delta x\rvert\, \Delta_\rangle P \geqslant \hbar$ (Arbatsky, 2005) $\Delta_\rangle (\, \delta x_1\, P_1 + \delta x_2\, P_2 + \delta x_3\, P_3\,) \geqslant \hbar$ (Arbatsky, 2005)
$\varphi,\ J$	$\Delta'_\rangle \Phi\, \Delta_\rangle J \geqslant \dfrac{\hbar}{2} \left[1 - \dfrac{3}{\pi^2} (\Delta'_\rangle \Phi)^2 \right]$ (Judge, 1964) $\delta_\rangle \Phi \geqslant \min\, (\, \hbar / \Delta_\rangle J\ ;\ \pi\,)$ (Arbatsky, 2006)	$\lvert\delta\varphi\rvert\, \Delta_\rangle J \geqslant \hbar$ (Arbatsky, 2005) $\Delta_\rangle (\, \delta\varphi_1\, J_1 + \delta\varphi_2\, J_2 + \delta\varphi_3\, J_3\,) \geqslant \hbar$ (Arbatsky, 2005)
$t,\ E$		$\lvert\delta t\rvert\, \Delta_\rangle H \geqslant \hbar$ (Mandelshtam, Tamm, 1945)
all	—	$\Delta_\rangle\, (\, -\delta x_\mu\, P_\mu + \frac{1}{2}\, \delta\omega_{\mu\nu}\, J_{\mu\nu}\,) \geqslant \hbar$ (Arbatsky, 2005)