On quantization of electromagnetic field.
III. Formula for generators of infinitesimal linear canonical transformations.

D. A. Arbatsky[*]

March, 2002

Abstract

Here we suggest a formula for generators of infinitesimal linear symplectic transformations of invariant phase space. We discuss applications of this formula to classical and quantum field theory. We show the existence of generators of the symmetry group for quantum case.

1. Formula. In the paper [I] we used a formula that gives generators of infinitesimal linear transformations for invariant Hamiltonian formalism. In view of importance and generality of this formula we will give here its derivation and discussion.

So, consider some linear field which is described by the invariant Hamiltonian formalism. Let us for definiteness look for the generators of the Poincare group.

So far as the symplectic structure ω of the invariant phase space is supposed to be Poincare-invariant, the Poincare group acts on Z as continuous linear symplectic group. For any infinitesimal transformation from this group[1] we have a Hamiltonian flow on Z . The Hamiltonian of this flow is a function on Z . Let us denote it $G^{\underline{\cdot}}$. Now we will find the explicit formula for this function.

First, it is immediately seen that this function is defined up to addition of a constant. We will fix this constant by the condition $G^{\underline{0}} = 0$, i. e. the function is equal to zero for the zero vector from space Z . Speaking in a more physical way, we will suppose that the vacuum of a classical field has zero energy, momentum etc.

Let us fix now in the space Z a point $\underline{c}$. The vector of velocity of the flow in this point we denote as $\underline{\delta c}$. Let us draw the straight line segment from the point $\underline{0}$ to the point $\underline{c}$, i. e. let us consider the set of points like $\alpha\,\underline{c}$, $\alpha \in [0;1]$. So far as transformations of the group under consideration are linear, in the point $\alpha\,\underline{c}$ the velocity of flow is equal to $\alpha\,\underline{\delta c}$. On the other hand, the vector of the velocity of flow is connected with the Hamiltonian by the relation:

$$dG\big|_{\alpha\underline{c}}{}^{\underline{b}} = \omega^{\underline{b}\,;\,\alpha\underline{\delta c}} \ , \qquad (1)$$

which is true for any $\underline{b} \in Z$. Here the left side denotes the differential of the function $G^{\underline{\cdot}}$ calculated in the point $\alpha\,\underline{c}$ and taken on the vector $\underline{b}$.

As a vector $\underline{b}$ in the equation (1) we can take the vector $d\alpha\,\underline{c}$:

$$dG\big|_{\alpha\underline{c}}{}^{d\alpha\underline{c}} = \omega^{\,d\alpha\underline{c}\,;\,\alpha\underline{\delta c}} \ .$$

Using bilinearity of the form ω we can write it as:

$$dG\big|_{\alpha\underline{c}}{}^{d\alpha\underline{c}} = \omega^{\underline{c}\,;\,\underline{\delta c}}\ \alpha\, d\alpha \ .$$

Integrating this relation with respect to α in the limits from 0 to 1 we get the formula for the generator:

$$\boxed{G^{\underline{c}} = \tfrac{1}{2}\,\omega^{\underline{c}\,;\,\underline{\delta c}}} \qquad (2)$$

[*]http://daarb.narod.ru/ , http://wave.front.ru/

[1]I. e. an element of its Lie algebra.

2. Application to classical field theory. It should not be thought that the formula (2) is just other form of writing the Noether formula. First, it works only in the linear case. Second, it is not supposed to be used for looking for *integrals of motion*, because they are supposed to be known for linear fields (the equations of the motion are solved). Nevertheless, the formula (2) turns out to be very convenient in practical application for many reasons.

First, it is applied much easier. Second, in contrast to the Noether formula, it is not connected with coordinate representation. If we know a formula for the symplectic structure ω, for example, in the Fourier-representation and we know how in this representations the group under consideration acts (for example, Poincare group) then we get the generators as easily as in the coordinate representation.

Furthermore, as it was shown in the paper [II], the structure of the invariant Hamiltonian formalism does not change, if we add to the Lagrangian a full divergence. So far as the formula (2) is written in the terms of this formalism only, the independence of generators from this substitution becomes apparent.

As a simplest example, consider the scalar field. In the paper [I] we have shown that in the Fourier representation the symplectic structure of this field is given by the formula:

$$\omega^{\,\underline{\cdot}\,;\,\underline{\cdot}} = \int d\mu_m \cdot i\,\varepsilon(k) \cdot a(-k)^{\,\underline{\cdot}}\, a(k)^{\,\underline{\cdot}} \;. \qquad (3)$$

Now consider a space-time shift of the field by infinitesimal vector $\delta\varepsilon_\nu$. A state $\underline{c}$ changes to $\underline{c} + \underline{\delta c}$. And:

$$a(k)^{\,\underline{\delta c}} = i\,\delta\varepsilon_\nu\, k_\nu\, a(k)^{\,\underline{c}} \;.$$

Substituting this expression to the formula (2), and taking into account the formula (3), we get:

$$G^{\,\underline{c}} = -\,\tfrac{1}{2}\,\delta\varepsilon_\nu \int d\mu_m \cdot \varepsilon(k) \cdot k_\nu\, a(-k)^{\,\underline{c}}\, a(k)^{\,\underline{c}} = \;-\,\delta\varepsilon_\nu \int d\mu_m^+ \cdot k_\nu\, a^*(k)^{\,\underline{c}}\, a(k)^{\,\underline{c}} \;.$$

Let us denote the coefficient of $\delta\varepsilon_\nu$ as $-P_\nu$. Omitting the argument $\underline{c}$ we get the formula for the vector of energy and momentum:

$$P_\nu = \tfrac{1}{2} \int d\mu_m \cdot \varepsilon(k) \cdot k_\nu\, a(-k)\, a(k) = \int d\mu_m^+ \cdot k_\nu\, a^*(k)\, a(k) \;. \qquad (4)$$

3. About generators in quantum field theory. In the paper [VI] we will describe how construction of quantum fields is performed. Let us discuss here how generators are found in the quantum case.

As far as I know, there was not suggested any analog of Noether theorem for the quantum field theory[2].

Note now, that in the case of invariant quantization of fields, in accordance with the scheme given in the paper [VI], the action of group of invariance is naturally transferred to the quantum space of states $\mathcal{H}$. And this action is unitary (in the case of indefinite scalar product — pseudounitary).

If the scalar product in $\mathcal{H}$ is positive-definite, then in accordance with the Stone theorem, we have the self-adjoint quantum generator. If the scalar product in $\mathcal{H}$ is indefinite, then everything is similar, but we need to introduce some special mathematical terminology.

So, generators of the group of invariance can be rigorously introduced in the quantum case also.

It seems to be of interest to get some useful formulas for quantum generators also.

So far as the vacuum $|\,0\,\rangle$ of the quantum field remains invariant under the action of the symmetry group, then, speaking formally, we just need to write expressions like (4) making normal ordering of operators in the integrand. In the case of Poincare-invariant quantization of the scalar field that leads to the positive-definite scalar product we get:

$$\hat{P}_\nu = \int d\mu_m^+ \cdot k_\nu\, \hat{a}^*(k)\, \hat{a}(k) \;. \qquad (5)$$

But, nevertheless, the strict mathematical meaning of the obtained expression is not fully clear[3]. It seems to be topical to develop a theory of integrals like (5) and prove that if the integrands are normally ordered, we get exactly the generators introduced here.

[2] Some authors allege opposite. I allow myself just to ignore their opinion on this question.

[3] It will be seen from discussion of quantization and topological questions in the paper [VI] that symbols like $\hat{a}^*(k)$ have sense only after a proper averaging. But in the given formula we need to integrate *the product* of such symbols.