On quantization of electromagnetic field.
II. Arbitrariness in choice of Lagrangian.

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Abstract

Here we show that addition to Lagrangian a divergence of a function does not change the symplectic structure on invariant phase space.

It is well-known that the same dynamics can be obtained by different choices of Lagrangian. The simplest replacement of this type is a linear substitution:

$$L \longrightarrow L' = \alpha L \ .$$

So far as in these papers we study the questions of quantization and normalize action by Planck's constant, $\hbar = 1$, this arbitrariness is absent in our research.

But there is another essential arbitrariness which is the addition to the Lagrangian a full divergence[1]:

$$L \longrightarrow L' = L + \partial_\mu F_\mu \ , \tag{1}$$

where F_μ is some vector function of the coordinate x and field values $\varphi_i(x)$.

In general case the Lagrangian may contain derivatives of the field with respect to coordinates of the order higher than first. Here we will take into account such a possibility and we will accept that the function F_μ may also depend on the derivatives of the field with respect to coordinates.

So far as the substitution (1) does not change dynamics, it leaves the invariant phase space Z unchanged. Now we will show that the symplectic structure on Z also remains unchanged. In fact, this means that the whole invariant Hamiltonian formalism does not feel such an addition. And so far as quantization is based on invariant Hamiltonian formalism, we get that the construction of the quantized field does not change under the substitution (1).

It seems to be difficult to prove the invariability of the symplectic structure using the formula for the symplectic current as a starting point. In this connection we will remind here how symplectic structure is defined directly in the variational terms [1, 2, 3, 4, 5, 6].

Consider the action in an arbitrary region Ω :

$$S = \int\limits_\Omega d^4x \, L(x, \varphi_i, \partial_\mu \varphi_i, \dots, \partial_{\mu_1} \dots \partial_{\mu_k} \varphi_i) \ . \tag{2}$$

Its variation, if the region is unchanged, is:

$$\delta S = \int\limits_\Omega d^4x \, \left(\frac{\delta L}{\delta \varphi_i} \, \delta\varphi_i + \partial_\mu j_\mu \right) \ , \tag{3}$$

Here

$$j_\mu = \frac{\delta L}{\delta(\partial_\mu \varphi_i)} \, \delta\varphi_i + \ldots + \frac{\delta L}{\delta(\partial_\mu \partial_{\mu_2} \dots \partial_{\mu_k} \varphi_i)} \, \partial_{\mu_2} \dots \partial_{\mu_k} \delta\varphi_i \ .$$

Variational derivative here is understood as

$$\frac{\delta}{\delta a} = \frac{\partial}{\partial a} - \partial_\mu \frac{\partial}{\partial(\partial_\mu a)} + \partial_\mu \partial_\nu \frac{\partial}{\partial(\partial_\mu \partial_\nu a)} - \dots \ .$$

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[1] In principle, there are other, more complicated substitutions which usually do not play a practical role because of their complexity.

The action (2) can be considered as a functional on the set of functions φ_i . Using the identity $\delta^2 = 0$ we get:

$$\int\limits_{\Omega} d^4x \left(\delta \left(\frac{\delta L}{\delta \varphi_i} \delta\varphi_i \right) + \partial_\mu \delta j_\mu \right) = 0 \ .$$

If we consider here variations only for functions φ_i for which action is stationary, then the first term in the last equation becomes zero. From here we get that the symplectic structure defined on Z as

$$\omega = \int\limits_{\Sigma} d\sigma_\mu \, \delta j_\mu(x)$$

does not depend on the choice of the space-like surface Σ (it is implied that Σ behaves well enough at the infinity).

Let us consider now how the above equations are changed under the substitution (1).

Variation of the new action S' can be written like in the formula (3):

$$\delta S' = \int\limits_{\Omega} d^4x \left(\frac{\delta L'}{\delta \varphi_i} \delta\varphi_i + \partial_\mu j'_\mu \right) = \int\limits_{\Omega} d^4x \left(\frac{\delta L}{\delta \varphi_i} \delta\varphi_i + \frac{\delta(\partial_\mu F_\mu)}{\delta \varphi_i} \delta\varphi_i + \partial_\mu j'_\mu \right) \ .$$

But $\delta(\partial_\mu F_\mu)/\delta\varphi_i = 0$. Therefore,

$$\delta S' = \int\limits_{\Omega} d^4x \left(\frac{\delta L}{\delta \varphi_i} \delta\varphi_i + \partial_\mu j'_\mu \right) \ . \tag{4}$$

On the other hand, this variation can be written in the terms of the old action:

$$\delta S' = \delta S + \int\limits_{\Omega} d^4x \, \partial_\mu \delta F_\mu \ . \tag{5}$$

Combining (4) and (5) with (3) we get:

$$\int\limits_{\Omega} d^4x \, (\partial_\mu j'_\mu - \partial_\mu j_\mu - \partial_\mu \delta F_\mu) = 0 \ .$$

So far as in this equation all three functions j'_μ , j_μ , and δF_μ are local (i. e. they become zero in regions where $\delta\varphi_i = 0$) we have:

$$\int\limits_{\Sigma} d\sigma_\mu \, (j'_\mu - j_\mu - \delta F_\mu) = 0 \ .$$

The value F_μ can be considered as a functional on the set of functions φ_i . Using the identity $\delta^2 = 0$ we get:

$$\int\limits_{\Sigma} d\sigma_\mu \, \delta j'_\mu(x) = \int\limits_{\Sigma} d\sigma_\mu \, \delta j_\mu(x) \ .$$

I. e. the substitution (1) leaves the symplectic structure on Z unchanged:

$$\omega' = \omega \ .$$

References

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