The certainty principle II
Proof of the uncertainty principle

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Abstract

A more detailed derivation of the Heisenberg uncertainty principle from the certainty principle is given.

Introductory remarks. After publication of the paper [1] I received many letters, including those with references. In this connection, I think it is necessary to specify the following:

1. The metric introduced by me ("quantum angle") is known to mathematicians from 1904 as *Fubini-Study metric.*
2. The correct "uncertainty relation" (in fact, *certainty* relation) for the quantities energy - time was first suggested by Mandelshtam and Tamm [2].
3. Mandelshtam and Tamm studied a quantum system in the Schrödinger representation and did not use group theory methods. For this reason they could not understand that their result has more general character.
4. In contrast, I used group theory formulations. And implied that the system can be studied, in particular, in the representation of relativistic canonical quantization [4]. This allowed me to formulate the certainty principle and to suggest more general inequalities.

Some of my critics refused to believe that the Heisenberg uncertainty principle is really a consequence of the certainty principle. They claimed that "this can not be so, because this can never be so".

Nevertheless, this is so. And here I give a more detailed explanation.

Derivation of the uncertainty principle. Suppose that for a given quantum system we succeeded to find some observable X , that can be considered in some sense a "coordinate operator".

Suppose that X is a self-adjoint operator with continuous spectrum, $X = X^*$. Let us denote $\Omega_{(a,b)}$ its spectral projector[1] for an arbitrary real interval (a, b) .

Suppose also that we have a self-adjoint operator P , $P = P^*$, such that for any a , b and δx we have equality:

$$e^{+\,i\,\delta x\;P/\hbar}\,\Omega_{(a+\delta x,b+\delta x)}\,e^{-\,i\,\delta x\;P/\hbar} \;=\; \Omega_{(a,b)} \;,$$

i. e. P is a "generator of spectral shifts" for X . As we know, such an operator is usually an operator of momentum.

Suppose now that the system is in state $\rangle$, $\langle\,|\,\rangle = 1$.

The quantity $\langle\,\Omega_{(a,b)}\,\rangle$, obviously, defines the probability to find the system inside the interval (a, b) . Let us define such l and r , that

$$\langle\,\Omega_{(-\infty,l)}\,\rangle \;=\; \langle\,\Omega_{(r,+\infty)}\,\rangle \;=\; \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-\cos^2 1} \;\approx\; 0,07926\ldots$$

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[1] Roughly speaking, spectral projector is an operator nullifying wave function in X -representation outside of the given interval.

It is easy to see, that l and r exist[2]. So, the quantity $\delta_\rangle X = r - l$ can be naturally called "uncertainty" of the coordinate X .

T h e o r e m. *The following inequality takes place (the uncertainty principle):*

$$\boxed{\delta_\rangle X \Delta_\rangle P \geqslant \hbar} \tag{1}$$

In order to prove this theorem let us first estimate the scalar product of the vector $\rangle$ and the shifted vector $e^{-i\,\delta_\rangle X\,P/\hbar}\rangle$:

$$\begin{aligned}
\big|\langle\,|\,e^{-i\,\delta_\rangle X\,P/\hbar}\,\rangle\big| &= \big|\langle\,(\,\Omega_{(-\infty,r)} + \Omega_{(r,+\infty)}\,)\,e^{-i\,\delta_\rangle X\,P/\hbar}\,\rangle\big| = \\
&= \big|\langle\,\Omega_{(-\infty,r)}\,e^{-i\,\delta_\rangle X\,P/\hbar}\,\rangle + \langle\,\Omega_{(r,+\infty)}\,e^{-i\,\delta_\rangle X\,P/\hbar}\,\rangle\big| \leqslant \\
&\leqslant \big|\langle\,\Omega_{(-\infty,r)}\,e^{-i\,\delta_\rangle X\,P/\hbar}\,\rangle\big| + \big|\langle\,\Omega_{(r,+\infty)}\,e^{-i\,\delta_\rangle X\,P/\hbar}\,\rangle\big| = \\
&= \big|\langle\,(\Omega_{(-\infty,r)})^2\,e^{-i\,\delta_\rangle X\,P/\hbar}\,\rangle\big| + \big|\langle\,(\Omega_{(r,+\infty)})^2\,e^{-i\,\delta_\rangle X\,P/\hbar}\,\rangle\big| = \\
&= \big|\langle\,\Omega_{(-\infty,r)}\,e^{-i\,\delta_\rangle X\,P/\hbar}\,\Omega_{(-\infty,l)}\,\rangle\big| + \big|\langle\,\Omega_{(r,+\infty)}\,e^{-i\,\delta_\rangle X\,P/\hbar}\,\Omega_{(l,+\infty)}\,\rangle\big| \leqslant \\
&\leqslant \langle\,\Omega_{(-\infty,r)}\,\rangle^{1/2}\,\langle\,\Omega_{(-\infty,l)}\,\rangle^{1/2} + \langle\,\Omega_{(r,+\infty)}\,\rangle^{1/2}\,\langle\,\Omega_{(l,+\infty)}\,\rangle^{1/2} = \\
&= \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-\cos^2 1}} \cdot \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-\cos^2 1}} + \\
&\qquad + \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-\cos^2 1}} \cdot \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-\cos^2 1}} = \cos 1 \,.
\end{aligned}$$

Here, coming from the fifth to the sixth line, we estimated both terms by the Cauchy-Bunyakovsky-Schwarz inequality.

So, for the quantum angle between the initial and the shifted vectors we have estimation:

$$\angle\big(\,\rangle\,,\ e^{-i\,\delta_\rangle X\,P/\hbar}\,\rangle\,\big) \ \geqslant\ 1\ .$$

But it means that under the action of $e^{-i\,\delta_\rangle X\,P/\hbar}$ the vector $\rangle$ changes *substantially* [1].

Applying the *certainty* principle [1], we directly get (1). ■

Discussion. Historically, it stacked up so [5] that now in all textbooks the Heisenberg uncertainty principle is illustrated with the help of the Kennard inequality,

$$\Delta_\rangle X\,\Delta_\rangle P \ \geqslant\ \frac{\hbar}{2}\ ,$$

which is not identical with (1). Its undoubted virtue is that its proof is easier to understand for a person, who just starts to study quantum mechanics. Nevertheless, I believe that (1) is more fundamental.

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References

[1] D. A. Arbatsky "The certainty principle" (2005),
http://daarb.narod.ru/tcp-eng.html , http://wave.front.ru/tcp-eng.html , arXiv:quant-ph/0506165 .
[Rus.: D. A. Arbatsky „Printsip opredelyonnosti" (2005),
http://daarb.narod.ru/tcp-rus.html , http://wave.front.ru/tcp-rus.html , arXiv:quant-ph/0506165 .]

[2] But, generally speaking, they are not unique. In order to eliminate this non-uniqueness, it is convenient to choose l maximum of the possible, and r — minimum. Then the distance $r - l$ will be minimum.

[2] L. I. Mandelshtam, I. E. Tamm, “The uncertainty relation between energy and time in nonrelativistic quantum mechanics”, J. Phys. (USSR) **9**, 249-254 (1945).
[Rus.: L. I. Mandelshtam, I. E. Tamm „Sootnoshenie neopredelyonnosti energiya-vremya v nerelyativistskoy kvantovoy mekhanike“, Izv. Akad. Nauk SSSR (ser. fiz.) **9**, 122-128 (1945), in [3], p. 306-315.]

[3] L. I. Mandelshtam, *Polnoe sobranie trudov*, v. 2, M.-L. (1947).

[4] D. A. Arbatsky “What is “relativistic canonical quantization”?” (2005), http://daarb.narod.ru/wircq-eng.html , http://wave.front.ru/wircq-eng.html , arXiv:math-ph/0503029 .
[Rus.: D. A. Arbatsky „Chto takoe „relyativistskoe kanonicheskoe kvantovanie“?“ (2005), http://daarb.narod.ru/wircq-rus.html , http://wave.front.ru/wircq-rus.html , arXiv:math-ph/0503029 .]

[5] J. Hilgevoord, J. Uffink “The uncertainty principle”, in *The Stanford encyclopedia of philosophy*, ed. E. N. Zalta (2001), http://plato.stanford.edu/archives/win2001/entries/qt-uncertainty/ .