The certainty principle

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Abstract

The notion of the quantum angle is introduced. The quantum angle turns out to be a metric on the set of physical states of a quantum system. Its kinematics and dynamics are studied. The *certainty* principle for quantum systems is formulated and proved. It turns out that the certainty principle is closely connected with the Heisenberg uncertainty principle (it presents, in some sense, an opposite point of view). But at the same time the certainty principle allows to give rigorous formulations for wider class of problems (it allows to rigorously interpret and ground the analogous inequalities for the pairs of quantities like time - energy, angle - angular momentum etc.).

The quantum angle and the certainty principle

The notion of the quantum angle. As it is known, the set of states of any quantum system forms a complex Hilbert space. We will denote it $\mathcal{H}$.

Elements of the space $\mathcal{H}$ we will denote as $a, b \ldots \in \mathcal{H}$.

The scalar product in $\mathcal{H}$ we will write as $\langle a|b\rangle$. It is linear with respect to the second argument and anti-linear with respect to the first one.

The norm of a vector a we will denote as $\|a\| = \langle a|a\rangle^{1/2}$.

Consider two non-zero vectors $a, b \in \mathcal{H}$. Let us define between them *the quantum angle* by the formula:

$$\angle(a,b) = \arccos \frac{\big|\langle a|b\rangle\big|}{\|a\|\,\|b\|} \ .$$

According to Cauchy-Bunyakovsky-Schwarz inequality, under the function arccos we have the value that is not greater than unity. Therefore, the quantum angle is a real number:

$$\angle(a,b) \in \mathbb{R} \ , \qquad 0 \leqslant \angle(a,b) \leqslant \frac{\pi}{2} \ .$$

For simplification of formulas we will later on always work with normalized vectors: $\|a\| = 1$, $\|b\| = 1$. In this case the formula for the angle is written simply as:

$$\angle(a,b) = \arccos \big|\langle a|b\rangle\big| \ .$$

Geometry of quantum angle. Let us consider the two extreme cases: $\angle(a,b) = 0$ and $\angle(a,b) = \pi/2$.

According to the Parseval equality, the first case takes place when the vectors differ only by phase factor:

$$\angle(a,b) = 0 \qquad \Longleftrightarrow \qquad a \parallel b \ .$$

From physical point of view, we can say that the corresponding quantum states are *identical*.

The second case takes place when vectors are orthogonal:

$$\angle(a,b) = \frac{\pi}{2} \qquad \Longleftrightarrow \qquad a \perp b \ .$$

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In this case we can say that the corresponding quantum states are *completely different*.

Implying this physical terminology, which is used in the considered extreme cases, let us introduce also the following definition. Let us say that the states described by the vectors a and b *differ not-substantially*, if $\angle(a,b) < 1$; let us also say that the states *differ substantially*, if $\angle(a,b) \geqslant 1$.

T h e o r e m. *For any three vectors $a, b, c \in \mathcal{H}$ the inequality takes place ("the triangle inequality"):*

$$\angle(a,c) \leqslant \angle(a,b) + \angle(b,c) \ .$$

In order to prove this theorem, let us first notice that, so far as the inequality is proved for three vectors, we can bound ourself with the case when the Hilbert space $\mathcal{H}$ is three dimensional: $\mathcal{H} = \mathbb{C}^3$.

Multiplying the vectors a , b , and c by appropriate factors and choosing orthonormal basis in $\mathcal{H}$ appropriately, we can achieve that components of these vectors take the form:

$$a = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} , \qquad b = \begin{pmatrix} b_1 & z & b_3 \end{pmatrix} , \qquad c = \begin{pmatrix} c_1 & c_2 & 0 \end{pmatrix} ,$$

where $c_1, c_2, b_1, b_3 \in [0;1]$ are real non-negative numbers, and $z \in \mathbb{C}$ is complex.

Let us introduce also the auxiliary vector b' :

$$b' = \begin{pmatrix} b_1 & |z| & b_3 \end{pmatrix} .$$

We have:

$$\angle(b',c) = \arccos\big(b_1 c_1 + |z|\, c_2\big) \leqslant \arccos|\, b_1 c_1 + z c_2\,| = \angle(b,c) \ .$$

So far as the three vectors a , b' , and c have real coordinates and unit lengths, the triangle inequality for them is the well known triangle inequality on the sphere in the real three-dimensional Euclidean space:

$$\angle(a,c) \leqslant \angle(a,b') + \angle(b',c) \ .$$

Combining the obtained inequalities we get:

$$\angle(a,c) \leqslant \angle(a,b') + \angle(b',c) = \angle(a,b) + \angle(b',c) \leqslant \angle(a,b) + \angle(b,c) \qquad \blacksquare$$

Summarizing what was said above, we can say that the quantum angle $\angle$, considered as a function on the set of pairs of physical quantum states (considered to phase factor), is a *metric*.

T h e o r e m. *The metric space of physical quantum states with the metric $\angle$ is complete.*

The proof of this theorem is not difficult, but it requires substantial technical work. So we omit it here.

Kinematics of quantum angle. Let now the vector r depend on the real parameter t : $t \in \mathbb{R}$, $r(t) \in \mathcal{H}$, $\|\, r(t)\,\| = 1$.

Let us define *the quantum velocity* $v(t)$ by the formula:

$$v(t) = \dot r(t) = \lim_{\delta t \to 0} \frac{r(t+\delta t) - r(t)}{\delta t} \ .$$

Let us define also *the quantum angular speed* $\omega(t)$ by the formula:

$$\omega(t) = \lim_{\delta t \to 0} \frac{\angle\big(\, r(t+\delta t)\, ,\, r(t)\,\big)}{|\,\delta t\,|} \ .$$

In order to express $\omega(t)$ through $v(t)$ let us decompose $v(t)$ into the two orthogonal components:

$$v_{\|}(t) = r(t)\, \langle r(t)|v(t)\rangle \ , \qquad v_{\perp}(t) = v(t) - v_{\|}(t) \ .$$

T h e o r e m. *The quantum angular speed is equal to the norm of the orthogonal component of the quantum velocity:*

$$\omega(t) = \|\, v_{\perp}(t)\,\| \ .$$

In order to prove this theorem let us use the Parseval equality to change arccos to arcsin :

$$\angle\big(\, r(t+\delta t)\,,\, r(t)\,\big) \;=\; \arccos \,\big|\, \langle\, r(t+\delta t)\,|\, r(t)\,\rangle \,\big| \;=$$
$$=\; \arcsin \,\big\|\, r(t+\delta t) - r(t)\,\langle\, r(t)\,|\, r(t+\delta t)\,\rangle \,\big\| \;=$$
$$=\; \arcsin \,\big\|\, r(t+\delta t) \;-\; r(t) \;-\; r(t)\,\langle\, r(t)\,|\, r(t+\delta t) - r(t)\,\rangle \,\big\| \;=$$
$$=\; \arcsin \,\big\|\, v(t)\,\delta t \;+\; o(\delta t) \;-\; r(t)\,\langle\, r(t)\,|\, v(t)\,\delta t + o(\delta t)\,\rangle \,\big\| \;=$$
$$=\; \arcsin \,\Big\|\, \big(v(t) \;-\; r(t)\,\langle\, r(t)\,|\, v(t)\,\rangle\big)\,\delta t \;+\; o(\delta t) \,\Big\| \;=$$
$$=\; \arcsin \,\big\|\, v_\perp(t)\,\delta t \;+\; o(\delta t) \,\big\| \;=\; \arcsin\,\big(\, \|v_\perp(t)\|\,|\delta t| \;+\; o(\delta t)\,\big) \;=$$
$$=\; \|v_\perp(t)\|\,|\delta t| \;+\; o(\delta t) \qquad \blacksquare$$

T h e o r e m. *The quantum angle satisfy the estimate:*

$$\angle\big(\, r(t_2)\,,\, r(t_1)\,\big) \;\leqslant\; \left|\; \int_{t_1}^{t_2} \omega(t)\, dt \;\right| \quad . \tag{1}$$

For the proof let us use the triangle inequality:

$$\Big|\angle\big(\, r(t+\delta t)\,,\, r(t_1)\,\big) - \angle\big(\, r(t)\,,\, r(t_1)\,\big)\Big| \;\leqslant\; \angle\big(\, r(t+\delta t)\,,\, r(t)\,\big)\;.$$

Dividing by $|\delta t|$ and taking the limit $\delta t \to 0$, we get:

$$\left|\; \frac{d}{dt}\angle\big(\, r(t)\,,\, r(t_1)\,\big) \;\right| \;\leqslant\; \omega(t)\;.$$

Performing integration from t_1 to t_2 , we get the desired inequality. $\blacksquare$

In fact, the estimate (1) is the best. Namely, there is the following

T h e o r e m. *The quantum angle between two vectors* r_1 *and* r_2 *can be expressed by the formula:*

$$\angle\,(\, r_2\,,\, r_1\,) \;=\; \min\, \left|\; \int_{t_1}^{t_2} \omega(t)\, dt \;\right| \quad ,$$

where the minimum is taken among all curves $r(t)$ *with ends in* r_1 *and* r_2 *:* $r(t_1) = r_1$ *,* $r(t_2) = r_2$ *.*

For the proof of the theorem let us twist the phase of r_2 , $r_2 \to r_2' = e^{i\alpha} r_2$, so that $\langle r_2' | r_1 \rangle$ become real and $\langle r_2' | r_1 \rangle \in [0,1]$.

Let us consider the linear shell of r_2' and r_1 , $\mathcal{L}(r_2', r_1)$. There we can choose an orthonormal basis so that

$$r_1 = (\;1 \quad 0\;)\,, \qquad r_2' = (\;a \quad b\;)\,,$$

where $a, b \in [0,1]$ are real non-negative numbers.

Considering then r_2' and r_1 as real vectors on Euclidean plane we see that it is possible to stretch the circular arc between them where the estimation integral is exactly equal to the quantum angle. $\blacksquare$

Dynamics of quantum angle. Let us have now a strongly continuous one-parameter unitary group $U(\delta s) = e^{-\,i\,\delta s\, A/\hbar}$, where $A = A^*$ is a self-adjoint operator in the space of states $\mathcal{H}$ (it is called the infinitesimal generator of $U(\delta s)$); $\delta s \in \mathbb{R}$ is the parameter of the group; $\hbar \in \mathbb{R}$ is the Planck's constant.

And suppose now that the dependence of state vector on the parameter δs is defined by the formula:

$$r(\delta s) \;=\; |\,\delta s\,\rangle \;=\; U(\delta s)\,\rangle \;=\; e^{-\,i\,\delta s\, A/\hbar}\,\rangle\;.$$

Here $|\,\delta s\,\rangle \in \mathcal{H}$ is another notation for the state vector connected with parameter equal to δs ; $\rangle \in \mathcal{H}$ is a fixed ket-vector of state.

Let us suppose that the function $r(\delta s)$ is differentiable. Then the quantum velocity is expressed by the formula:

$$v(\delta s) \;=\; \tfrac{1}{i\hbar}\, A\, e^{-\,i\,\delta s\, A/\hbar}\,\rangle \;=\; \tfrac{1}{i\hbar}\, A\,|\,\delta s\,\rangle\;.$$

The mean of the operator A does not depend on time:

$$\overline{A} = \langle\, \delta s \,|\, A \,|\, \delta s \,\rangle = \langle\, e^{+\,i\,\delta s\,A/\hbar}\, A\, e^{-\,i\,\delta s\,A/\hbar} \,\rangle = \langle\, A\, e^{+\,i\,\delta s\,A/\hbar}\, e^{-\,i\,\delta s\,A/\hbar} \,\rangle = \langle\, A \,\rangle\ .$$

Therefore the components of the quantum velocity can be written just as:

$$v_{\|}(\delta s) = |\, \delta s \,\rangle \langle\, \delta s \,|\, \tfrac{1}{i\hbar}\, A \,|\, \delta s \,\rangle = \tfrac{1}{i\hbar}\, \overline{A} \,|\, \delta s \,\rangle$$

$$v_{\perp}(\delta s) = \tfrac{1}{i\hbar}\, \big(A - \overline{A}\big)\, |\, \delta s \,\rangle\ .$$

The quantum angular speed turns out to be independent of the parameter also:

$$\omega(\delta s) = \|\, v_{\perp}(\delta s) \,\| = \tfrac{1}{\hbar}\, \langle\, \delta s \,|\, \big(A - \overline{A}\big)^2 \,|\, \delta s \,\rangle^{1/2} = \tfrac{1}{\hbar}\, \langle\, e^{+\,i\,\delta s\,A/\hbar}\, \big(A - \overline{A}\big)^2\, e^{-\,i\,\delta s\,A/\hbar} \,\rangle^{1/2} =$$

$$= \tfrac{1}{\hbar}\, \langle\, \big(A - \overline{A}\big)^2\, e^{+\,i\,\delta s\,A/\hbar}\, e^{-\,i\,\delta s\,A/\hbar} \,\rangle^{1/2} = \tfrac{1}{\hbar}\, \langle\, \big(A - \overline{A}\big)^2 \,\rangle^{1/2} = \tfrac{1}{\hbar}\, \Delta_{\rangle} A\ .$$

Here $\Delta_{\rangle} A$ is a short notation for the standard deviation of A in the state $\rangle$.

Consider now how the quantum angle $\angle(\,|\, \delta s \,\rangle\,,\, \rangle\,)$ behaves in this case. Using for it the estimate (1) we have:

$$\angle(\,|\, \delta s \,\rangle\,,\, \rangle\,) \leqslant \left|\, \int_0^{\delta s} \omega(\sigma)\, d\sigma \,\right| = \tfrac{1}{\hbar}\, |\delta s|\, \Delta_{\rangle} A\ .$$

From this inequality we obtain the

T h e o r e m. *So that under the action of strongly continuous one-parameter unitary group* $U(\delta s) = e^{-\,i\,\delta s\,A/\hbar}$ *the initial state vector* $\rangle$ *changes substantially, it is necessary to satisfy the inequality:*

$$\boxed{\ |\delta s|\, \Delta_{\rangle} A \geqslant \hbar\ } \tag{2}$$

By the example of the Schrödinger particle we will see that this theorem turns out to be closely connected with the Heisenberg uncertainty principle, but has other meaning. Taking into account this connection, we can name this theorem *the certainty principle*.

The inequality expressing the certainty principle can be written also in the following way:

$$\boxed{\ \Delta_{\rangle}(\delta s A) \geqslant \hbar\ } \tag{3}$$

In this form it can be naturally carried over to the case when δs and A are matrices.

Examples

One-dimensional Schrödinger particle. Consider the one-dimensional Schrödinger particle with the coordinate defined by the variable x . Its state vector can be written by the wave function $\psi(x)$. In its space of states the group of shifts acts by the formula:

$$U(\delta x)\, \psi(x) = \psi(x - \delta x)\ .$$

This group can be written in the form:

$$U(\delta x) = e^{-\,i\,\delta x\,P/\hbar}\ , \qquad P = -\,i\hbar \frac{d}{dx}\ .$$

Here P is the operator of momentum.

Applying the certainty principle in the form (2), we get:

$$|\delta x|\, \Delta_{\psi(x)} P \geqslant \hbar\ .$$

If we take as $\psi(x)$ a well localized packet of de Broglie waves, that turns into zero outside of some interval l , then from this inequality, in particular, we have that $l \geqslant \hbar / \Delta_{\psi(x)} P$: because when the packet is moved to the distance l , the change of the quantum angle must turn out to be greater than 1 (namely, $\pi/2$).

So, the Heisenberg uncertainty principle, if it is understood in qualitative sense, follows from the certainty principle.

But if we understand the Heisenberg uncertainty principle in quantitative sense, according to the Pauli-Weyl inequality

$$\Delta_{\psi(x)} X \, \Delta_{\psi(x)} P \geqslant \frac{\hbar}{2} \ , \tag{4}$$

then there is no direct connection between these two principles.

Furthermore, from physical point of view, the Heisenberg uncertainty principle and the certainty principle are like two points of view on the spread of the wave packet. The Heisenberg uncertainty principle says, that the wave packet is spread, because the *classical* state of the particle is *badly defined*. On the other hand, the certainty principle states, that the wave packet is spread, because the *quantum* state is *well defined*.

To the three-dimensional case the certainty principle is easily generalized in the form (3):

$$\Delta_{\psi(x)} \left(\delta x_i \, P_i \right) \geqslant \hbar \ ,$$

where summation over i is implied.

Schrödinger particle on circle. Consider now a plane with Cartesian coordinates x and y. Let us have the circle $x^2 + y^2 = 1$ defined on the plane. On this circle as one-dimensional coordinate we can use the polar angle φ, considered to 2π.

The state vector of the Schrödinger particle on this circle can be written by the wave function $\psi(\varphi)$. And $\psi(\varphi + 2\pi) = \psi(\varphi)$.

In the space of states the action of the rotation group is naturally defined:

$$U(\delta\varphi) \, \psi(\varphi) = \psi(\varphi - \delta\varphi) \ .$$

This group can be written in the form:

$$U(\delta\varphi) = e^{-\, i \, \delta\varphi \, J/\hbar} \ , \qquad J = -\, i\hbar \frac{d}{d\varphi} \ .$$

Here J is the operator of angular momentum.

Using the certainty principle does not arise any difficulties:

$$|\delta\varphi| \, \Delta_{\psi(\varphi)} J \geqslant \hbar \ .$$

As regards the uncertainty principle, a carrying over of the inequality (4) to this case is impossible[1].

In the three-dimensional case the certainty principle also easily gives:

$$\Delta_{\psi(x,y,z)} \left(\delta\varphi_i \, J_i \right) \geqslant \hbar \ .$$

A system with Hamiltonian independent of time. Let us have now a quantum system with a Hamiltonian H independent of time. On the space of states we have the following action of the group of time shifts:

$$U(\delta t) \, \rangle = e^{-\, i \, \delta t \, (-H)/\hbar} \rangle \ .$$

The certainty principle in this case immediately gives:

$$|\delta t| \, \Delta_{\rangle} H \geqslant \hbar \ .$$

If we apply this formula, for example, to estimation of the life time of a quasi-stationary decaying state, then it states that its typical life time is not less than Planck's constant divided by the width of the corresponding energy level. And here all the terms can be defined with exact mathematical sense.

As regards to attempts to formulate the Heisenberg uncertainty principle for values time - energy, after Bohr has declared such a principle (in qualitative sense) so many researches and discussions were performed for clarification of its sense, that it is possible to write about them a separate review. As far as I know, a rigorous formulation of the uncertainty principle for this case have not been formulated till now[2].

[1] About some attempts that were made to suggest an inequality like (4), see [1].
[2] See also the discussion of this question by J. Baez [2].

Relativistic systems. Consider now any relativistic quantum system. On its space of states the Poincare group acts. As an example of such a system any RCQ-quantized field can serve[3] [3, 4]. And let us restrict ourself to the discussion of the case when the field turns out quantized in the usual Hilbert space.

In this case the application of the certainty principle does not meet any difficulties:

$$\Delta_{\rangle} \left(-\, \delta x_\mu \, P_\mu \, + \, \tfrac{1}{2} \, \delta\omega_{\mu\nu} \, J_{\mu\nu} \, \right) \geqslant \hbar \; ,$$

where P_μ is the vector operator of energy-momentum, $J_{\mu\nu}$ is the tensor operator of four-dimensional angular momentum, δx_μ and $\delta\omega_{\mu\nu}$ are the standard logarithmic coordinates of the Poincare group.

As regards to the application of the Heisenberg uncertainty principle, it is unlikely to be possible. In the previous paragraph we have seen that for the values time - energy it arose great difficulties.

In this connection, even from the ideas of relativistic invariance it is clear that even for coordinates and momenta the situation cannot be simple. And it turns out to be so, because it is known that all attempts to introduce the notion of coordinates (as self-adjoint operators on the space of states) for relativistic systems are quite artificial[4].

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[3] All other examples that I know either come from this or are mathematically insolvent.

[4] And it turns out that for some systems introduction of good operators of coordinates is not possible at all [5].