On quantization of electromagnetic field.
VI. Quantization.

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Abstract

Here we describe a general method of quantization of linear fields. We introduce a conception of quantization invariant with respect to action of some group. A space of quantum states of relativistic fields is constructed in apparently relativistic-invariant way. A connection with quantization of the field oscillator is established. We substantiate the necessity of using an indefinite scalar product for electromagnetic field. We discuss additional condition for "physically allowed" states of electromagnetic field. We discuss properties of the space of states of electromagnetic field from the point of view of functional analysis. We consider the question about origin of anti-unitary transformations in quantum field theory.

1. Other approaches to quantization. Before we start description of construction of a quantized field, let us give here a brief comparison of our construction with other approaches to description of quantized fields.

- **Method of formal manipulations.** This method was used even in the earliest papers on quantum field theory. Its essence is that we calculate Poisson brackets for classical values and accept that for corresponding quantum values commutators can be calculated by the formula:

$$[\,\hat{a}\,,\,\hat{b}\,] = i\,\{\,a\,,\,b\,\}^{\widehat{\ }}\,. \tag{1}$$

  Of course, this formula can not be true for all values. Nevertheless, if we use it cautiously enough, we can ascertain many properties of fields, even without clarifying what they are. In fact, this approach turned out to be the most effective from the point of view of practical applications.

  With regards to how Poisson brackets were calculated, it was performed in such a way that it was very difficult to see relativistic invariance even in the classical theory. Later Peierls [1] have suggested an alternative method of obtaining commutation relations, which was apparently relativistic-invariant. The corresponding classical bracket used in the expression (1), was even named "the Peierls bracket". A quite complicated "physical" grounding of the relations (1), used for quantization in the Peierls method, can be found in [2].

  The approach described is most close to ours.

  About classical description of relativistic fields we can say that invariant Hamiltonian formalism allows to define the Poisson bracket in apparently relativistic-invariant way. In this approach the notion of the Poisson bracket changes in such a way that the Peierls bracket turns out to be its very particular case. Methods of practical calculations of Poisson brackets were described by me in the paper [I].

  With regards to quantization of fields, in this paper in will be performed *constructively*: on the base of the developed in [I] invariant Hamiltonian formalism, using usual algebraic methods, we will construct in this paper quantum fields evidently.

- **Method of box.** This method was also very popular in the field theory from its birth. The essence of it is that we consider field not in the infinite space, but in a large box with periodical boundary conditions. So, the description of the field becomes similar to the description of infinite number of oscillators.

  The introduction of a box breaks the relativistic invariance in essence, and this reason is sufficient to not consider this method in detail.

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Nevertheless, as a result of using this method a useful notion of the field oscillator appeared. In this connection we should notice that in our method we have an analogous notion (introduced in the paper [IV]). We introduce it in a somewhat more abstract way with the use of the notion of an induced symplectic representation. Only by this way it is possible to ascertain the correct group-theory nature of the field oscillator and to connect invariant quantizations of field with invariant quantizations of oscillator.

- **Fock construction and Wigner-Mackey theory.** In the paper [3] Fock suggested that the space of states of a quantum field is a symmetrized tensor exponent of one-particle subspace. Later Wigner and Mackey developed a method of searching for all possible one-particle subspaces, based on the theory of induced unitary representations of Poincare group (see, for example, a review [5]). A quite detailed account of this approach is given in the book [7].

  The shortcomings of this approach are the following. First, operators of creation and destruction of quanta are introduced with quite complicated formulas from the very beginning. Second, even after introducing of these operators quite complicated reasoning is necessary for construction of local operators of *field*. Third, in this approach we can not consider fields with indefinite scalar product at all.

  In our approach the initial object is not a one-particle quantum space, but it is a classical field described by the language of the invariant Hamiltonian formalism. So, the second shortcoming disappears, because field operators are constructed during quantization.

  Operators of creation and destruction of quanta we can also introduce. And formulas for them are just the consequence of our algebraic construction of quantum field, and they are not postulated from the beginning.

  With regards to quantum fields with indefinite scalar product, in our approach they appear as naturally as fields with positive-definite scalar product. Our scheme is really more general, and it can be seen even with the example of massive scalar field. Due to this generality, the scheme naturally includes electromagnetic field as a particular case.

  Let us notice also that in our approach there is also an analog of Wigner-Mackey theory. This is the theory of symplectic representations of Poincare group [IV]. The theory of induced symplectic representations is richer than the theory of induced unitary representations.

- **Approach connected with Stone-von Neumann theorem.** In quantum mechanics when a quantum oscillator is studied the Stone-von Neumann theorem turns out to be useful; it says that canonical commutation relations for coordinates and momenta uniquely define corresponding irreducible unitary representation. Of course, people tried to bring this approach to quantum field theory.

  But the Stone-von Neumann theorem can not be directly changed for the case of quantization of infinite dimensional systems (see, for example, [8]). Great researches were made to overcome this difficulty. Despite of their undoubted importance, I still think that they turned out to be too far from practical applications.

  It should be noticed here also that if we do not require positive-definiteness of scalar product in the space of states, the uniqueness of quantization is lost even in the case of one-dimensional harmonic oscillator. Therefore, our approach differs from this essentially: we refuse to require positive-definiteness of scalar product, but instead we introduce the conception of invariant quantization.

  Let us mention here the following fact. When the Stone-von Neumann theorem is formulated, it is necessary to write commutation relations in the form of Weyl relations. But in our approach we use commutation relations just for unbounded operators. And the construction of quantization rigorously define in what sense they must be understood.

- **Geometric quantization.** The method of geometric quantization is that using geometry of the classical phase space we construct the space of quantum states as a some set of functions on classical phase space (see, for example, [9]).

  From the point of view of the invariant Hamiltonian formalism it is natural to apply this scheme to invariant phase space Z . But let us notice that substantial difficulties are seen immediately for this approach. First, it is not simple at all to make this approach mathematically rigorous for systems with infinite number of degrees of freedom. Second, the connection with Fock construction in this case is given by quite non-evident formulas. For these reasons this approach turns out to be not very convenient for practical applications. Third, it is not clear how we can describe by this approach the case of quantization with indefinite scalar product: even in the case of one-dimensional harmonic oscillator great difficulties arise with definition of indefinite scalar product.

- **Method of path integral.** This method became very popular now because it is supposed to be apparently relativistic-invariant. In fact, people work with path integral as with a symbolic form of writing of some expressions.

On the other hand, it is possible to consider path integral as rigorously defined mathematical object (see, for example, [10]). But such an approach meets so substantial difficulties that we can not talk about any "apparent" relativistic invariance at all.

2. Quantization. Let us describe now, how having classical field we construct quantum field.

1. Consider the set of (complex) functions that are constant on the whole space Z. Let us denote this set as $\mathcal{C}$. Consider also the direct sum $\mathcal{C} \oplus Z^*_{\mathbb{C}}$. The Poisson bracket turns this sum into a complex Lie algebra.

2. Disengaging now from the algebraic structure of the set $\mathcal{C} \oplus Z^*_{\mathbb{C}}$ and considering all its elements as completely independent let us call this set an *alphabet* $\mathcal{A}$ and its elements *letters*. Let us make an agreement that an element of $\mathcal{C} \oplus Z^*_{\mathbb{C}}$ considered as a letter of the alphabet $\mathcal{A}$ will be written additionally with a "hat" on top or on side, for example: $\hat{a} = a\widehat{\ }$.

3. Furthermore, we can introduce a formal product of elements of the alphabet. Namely, let us suppose that if we multiply a (finite) set of letters we get a *word* consisting of these letters (the order of letters is important here):
$$\hat{a}_1 \cdot \hat{a}_2 \cdot \ldots \cdot \hat{a}_k = \hat{a}_1 \hat{a}_2 \ldots \hat{a}_k \ , \qquad \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_k \in \mathcal{A} \ .$$
Words we will also denote by symbols with the hat, for example: $\hat{w} = w\widehat{\ }$. Let us denote the set of all possible words by the symbol $\mathcal{W}$; it is convenient to include in this set the word of zero length that we can denote as $()\widehat{\ }$. It is natural to accept that the operation of multiplication works on the set of words $\mathcal{W}$ also, and it "joins" words. So, $\mathcal{A}$ is a *system of free generators*, and $\mathcal{W}$ is a *free semi-group with neutral element* (the role of neutral element is played by the special element $()\widehat{\ }$).

4. Similarly, let us define the formal sum of (finite) set of words multiplied by arbitrary complex factors. Namely, let us call a *phrase* a formal expression like:
$$\lambda_1 \hat{w}_1 + \lambda_2 \hat{w}_2 + \ldots + \lambda_m \hat{w}_m \ , \qquad \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{C} \ , \quad \hat{w}_1, \hat{w}_2, \ldots, \hat{w}_m \in \mathcal{W} \ .$$
Let us also suppose two phrases to be equivalent, if for every word they have the same sum of factors for this word. The set of all such equivalence classes we will denote with the symbol $\mathcal{P}$.
On the set of phrases $\mathcal{P}$ we can now naturally introduce operations of addition and multiplication by a scalar. Namely, the sum of phrases is the phrase that arise if we join initial phrases by the symbol " + ". Multiplication of a phrase by a number is defined as multiplication of all factors by this number. So, $\mathcal{P}$ gets natural structure of complex linear space ($\mathcal{P}$ is a *free* $\mathbb{C}$ *-module generated by* $\mathcal{W}$).
On the set of phrases $\mathcal{P}$ we can also naturally introduce the operation of multiplication of two elements. Namely, the product of two phrases, each of which contains one word, we define by equality:
$$(\lambda_1 \hat{w}_1) \cdot (\lambda_2 \hat{w}_2) = (\lambda_1 \lambda_2)\, (\hat{w}_1 \hat{w}_2) \ , \qquad \lambda_1, \lambda_2 \in \mathbb{C} \ , \quad \hat{w}_1, \hat{w}_2 \in \mathcal{W} \ .$$
And we spread this definition to arbitrary phrases by the requirement of distributivity:
$$\hat{p}_1 \cdot (\hat{p}_2 + \hat{p}_3) = \hat{p}_1 \hat{p}_2 + \hat{p}_1 \hat{p}_3 \ , \qquad (\hat{p}_1 + \hat{p}_2) \cdot \hat{p}_3 = \hat{p}_1 \hat{p}_3 + \hat{p}_2 \hat{p}_3 \ , \qquad \hat{p}_1, \hat{p}_2, \hat{p}_3 \in \mathcal{P} \ .$$

5. So, the set of phrases $\mathcal{P}$ becomes an (associative) algebra ($\mathcal{P}$ is a *semi-group algebra of the semi-group* $\mathcal{W}$ or a *free algebra over semi-group* $\mathcal{W}$).

6. Up to now when we constructed the algebra $\mathcal{P}$ the alphabet $\mathcal{A}$ was considered as a completely arbitrary set. Now, using structure of Lie algebra in $\mathcal{C} \oplus Z^*_{\mathbb{C}}$ and the correspondence between the subspace $\mathcal{C}$ and the field of scalars $\mathbb{C}$ we can additionally bring to the algebra $\mathcal{P}$ some *defining relations*. It is performed by factorization of the algebra $\mathcal{P}$ with respect to the appropriate ideal.
Namely, consider all phrases of the following types:
$$(\lambda\, a)\widehat{\ } - \lambda\, \hat{a} \ ,$$
$$(a + b)\widehat{\ } - (\hat{a} + \hat{b}) \ ,$$
$$\{\, a \, , \, b \,\}\widehat{\ } + i\, (\hat{a}\, \hat{b} - \hat{b}\, \hat{a}) \ ,$$
$$\hat{1} - ()\widehat{\ } \ .$$
Here $\lambda \in \mathbb{C}$, $a, b \in \mathcal{C} \oplus Z^*_{\mathbb{C}}$, $\hat{a}, \hat{b} \in \mathcal{A}$. 1 — is a notation for the function that is equal to 1 on the whole invariant phase space Z, i. e. $1 \in \mathcal{C} \oplus Z^*_{\mathbb{C}}$; $\hat{1}$ — the element of the alphabet that corresponds to it, $\hat{1} \in \mathcal{A}$.

Let us span now on these phrases the two-sided ideal, i. e. we join to them the phrases that can be made from the given by using finite number of operations of multiplication by a factor, addition, and multiplication (from the left and the right) by arbitrary phrases. Factorizing the algebra $\mathcal{P}$ with respect to the obtained ideal we get an algebra that we will call an *algebra of operators* and will denote as $\mathcal{O}$. The elements of this algebra are called *operators* and are denoted as phrases corresponding to them.

The performed factorization leads to the result that in the algebra of operators we have the following defining relations:

$$(\lambda\, a)\widehat{\ } = \lambda\, \hat{a}\ ,$$

$$(a + b)\widehat{\ } = \hat{a} + \hat{b}\ ,$$

$$\{\, a\,,\, b\,\}\widehat{\ } = -\, i\, [\, \hat{a}\,,\, \hat{b}\,]\ ,$$

$$\hat{1} = ()\widehat{\ }\ .$$

In the third relation we have used the standard notation for *commutator* of two operators: $[\, \hat{a}\,,\, \hat{b}\,] = \hat{a}\, \hat{b} - \hat{b}\, \hat{a}$.

The fourth relation, in particular, allows us to escape using the cumbersome notation $()\widehat{\ }$, when we talk about operator and to write always $\hat{1}$.

7. Let us introduce now the operation of *conjugation*. This operation will be always denoted by the symbol * , without distinction of an object that it is applied to.

   Under conjugation of a complex number we will imply the usual complex conjugation.

   Elements of the space $\mathcal{C} \oplus Z^*_{\mathbb{C}}$ are functions on Z . Under conjugation applied to such a function we will imply a transition to the function with complex-conjugate values. Obviously, in the result we also get an element of the space $\mathcal{C} \oplus Z^*_{\mathbb{C}}$. Furthermore, so far as the Poisson bracket in $\mathcal{C} \oplus Z^*_{\mathbb{C}}$ is a complexified real Poisson bracket, we have the equality:

$$\{\, a^*\,,\, b^*\,\} = \{\, a\,,\, b\,\}^*\ . \qquad (2)$$

   The definition of conjugation can be naturally spread to the elements of the alphabet $\mathcal{A}$:

$$(\hat{a})^* = (a^*)\widehat{\ }\ .$$

   To words and to phrases the operation of conjugation is spread by the rules:

$$(\hat{a}\, \hat{b})^* = \hat{b}^*\, \hat{a}^*\ , \qquad (\lambda\, \hat{a})^* = \lambda^*\, \hat{a}^*\ , \qquad (\hat{a} + \hat{b})^* = \hat{a}^* + \hat{b}^*\ .$$

   Here $\hat{a}$ and $\hat{b}$ can be letters, words, and phrases; λ is a complex number.

   Obviously, the operation of conjugation is correctly spread from phrases to operators.

8. Consider now the space $Z^*_{\mathbb{C}}$. Let us split it into a direct sum of two subspaces:

$$Z^*_{\mathbb{C}} = Cr \oplus De\ .$$

   Let us require Cr and De , first, to come to each other under conjugation:

$$Cr^* = De\ . \qquad (3)$$

   Second, let us require that the Poisson bracket on these subspaces be zero[1]:

$$\{\, a\,,\, b\,\} = 0\ , \qquad a, b \in Cr\ ;$$

$$\{\, a\,,\, b\,\} = 0\ , \qquad a, b \in De\ .$$

   The subspace Cr we will call *creating*, and De — *destructing*. In accordance with the construction given above, for subspaces Cr and De we have some corresponding subspaces in the algebra of operators; let us denote these subspaces $Cr\widehat{\ }$ and $De\widehat{\ }$, correspondingly.

   Let us introduce also a notation $\mathcal{C}\widehat{\ }$ for the subspace of the algebra of operators that corresponds to the subspace $\mathcal{C}$ of the Lie algebra $\mathcal{C} \oplus Z^*_{\mathbb{C}}$.

[1] Taking into account (2) and (3), it is enough to require that it is zero on one of these subspaces.

9. Let us span now the left ideal on the subspace $De^\frown$, i. e. let us join to operators there operators that are obtained from them by finite number of operations of multiplication by scalar, addition, and multiplication from the left with arbitrary elements of the algebra $\mathcal{O}$.

   Let us factorize now the algebra $\mathcal{O}$ with respect to the constructed left ideal. In result we get some linear space. It is not an algebra, but it is a left module, i. e. there we have a naturally defined action of operators from the left. Elements of this factor-space are called *ket-vectors*. Ket-vectors are denoted with the symbols like $|\,x\,\rangle$, where x is any symbol describing the given vector. The space of ket-vectors is denoted $\mathcal{H}$ and is called *the space of states* of the quantized field.

10. We can perform the same construction with conjugate objects, i. e. instead of the subspace $De^\frown$ we can consider the subspace $Cr^\frown$, span on it the right ideal instead of left, and, factorizing, get the right module instead of left. The elements of this module are called *bra-vectors*. And this module we will call, if it does not make a misunderstanding, also *the space of states* of the quantized field, and we will denote it also as $\mathcal{H}$.

    Bra-vectors are denoted by symbols like $\langle\,x\,|$. The operation of conjugation is naturally spread to ket- and bra-vectors and defines the one-to-one correspondence between them. It is said that the ket- and bra-vectors, corresponding to each other, belong to the same state of the quantized field.

11. When we factorize the algebra of operators $\mathcal{O}$ and come to the spaces of ket- and bra-vectors, the operator $\hat{1}$ turns into some ket-vector $|\,0\,\rangle$ and bra-vector $\langle\,0\,|$, correspondingly. These ket- and bra-vectors describe the state which is called *vacuum*.

12. Let us introduce now the so called *scalar product* of bra- and ket- vectors. For an arbitrary bra-vector $\langle\,x\,|$ and a ket-vector $|\,y\,\rangle$ this product is written as $\langle\,x\,|\cdot|\,y\,\rangle$, or, for shortness, $\langle\,x\,|\,y\,\rangle$. As a result we get a complex number, i. e. $\langle\,x\,|\,y\,\rangle\in\mathbb{C}$.

    Let us require that the operation of scalar product would satisfy the following properties:

$$\langle\,0\,|\,0\,\rangle = 1\ ,$$

$$\big(\lambda\,\langle\,x\,|\big)\cdot|\,y\,\rangle = \lambda\cdot\langle\,x\,|\,y\,\rangle\ , \qquad \big(\langle\,x_1\,|+\langle\,x_2\,|\big)\cdot|\,y\,\rangle = \langle\,x_1\,|\,y\,\rangle+\langle\,x_2\,|\,y\,\rangle\ ,$$

$$\langle\,x\,|\cdot\big(\lambda\,|\,y\,\rangle\big) = \lambda\cdot\langle\,x\,|\,y\,\rangle\ , \qquad \langle\,x\,|\cdot\big(|\,y_1\,\rangle+|\,y_2\,\rangle\big) = \langle\,x\,|\,y_1\,\rangle+\langle\,x\,|\,y_2\,\rangle\ ,$$

$$\big(\langle\,x\,|\,\hat{o}\big)\cdot|\,y\,\rangle = \langle\,x\,|\cdot\big(\hat{o}\,|\,y\,\rangle\big)\ .$$

    In the latter equality $\hat{o}\in\mathcal{O}$; this property allows to write in such cases simply: $\langle\,x\,|\,\hat{o}\,|\,y\,\rangle$.

    The given properties for the scalar product are defining, i. e. there is not more than one scalar product satisfying the given properties.

13. The scalar product is a Hermitian form in $\mathcal{H}$, i. e. always $\langle\,x\,|\,y\,\rangle = (\langle\,y\,|\,x\,\rangle)^*$. If the scalar product turns out to be positive-definite, it defines in $\mathcal{H}$ some topology. If we complete $\mathcal{H}$ with respect to this topology, it turns into a usual Hilbert space. And operators appear to be defined on the corresponding dense linear subspace in $\mathcal{H}$.

    But the scalar product can be not positive-definite at all: it depends on the Poisson bracket in $\mathcal{C}\oplus Z^*_{\mathbb{C}}$ and the choice of the subspaces Cr and De . The question about definition of topology in this case for example of electromagnetic field will be discussed in the section 10.

So, in this section we have described how, if we have the Lie algebra of observables of a classical field $\mathcal{C}\oplus Z^*_{\mathbb{C}}$, we construct the algebra $\mathcal{O}$ of operators of the quantum field and the space of states $\mathcal{H}$. This construction is called *quantization* of a classical field.

3. Graduation of algebra $\mathcal{O}$ and of space $\mathcal{H}$. Connection with Fock construction. Consider now in the algebra of operators the subspaces $\mathcal{C}^\frown$, $Cr^\frown$ and $De^\frown$. Let us call them subspaces of the *grade* 0 , $+1$ and -1 , correspondingly. Making all possible products of these operators, let us assign to each such product a grade that is equal to the sum of grades of factors. If two products have the same grade, let us assign the same grade to their sum. It is easy to see that the algebra of operators, considered as a linear space, can be represented as a sum of its linear subspaces belonging to different grades:

$$\mathcal{O} = \ldots\oplus\mathcal{O}_{-2}\oplus\mathcal{O}_{-1}\oplus\mathcal{O}_0\oplus\mathcal{O}_{+1}\oplus\mathcal{O}_{+2}\oplus\ldots$$

And if $\hat{a}\in\mathcal{O}_i$ and $\hat{b}\in\mathcal{O}_j$, then $\hat{a}\,\hat{b}\in\mathcal{O}_{i+j}$. Shortly speaking, the algebra $\mathcal{O}$ is graded.

This graduation is naturally transferred to the space of states $\mathcal{H}$. And negative grades in the space $\mathcal{H}$ correspond to trivial subspaces. In other words, the space of states $\mathcal{H}$ is positive-graded:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \tag{4}$$

If a ket-vector belongs to one of the subspaces $\mathcal{H}_n$, we say that there are n particles in this state.

It is obvious also, that all this construction is possible also for conjugate objects, and decomposition (4) looks exactly the same for bra-vectors.

It is not out of place to mention here that, according to the introduced definition, the number of particles is always non-negative, and it does not depend on whether or not the scalar product in $\mathcal{H}$ is positive-definite.

It easy to see also that subspaces $\mathcal{H}_i$ and $\mathcal{H}_j$ are orthogonal with respect to the scalar product. In other words, if $\langle\, a\,| \in \mathcal{H}_i$ and $|\, b\,\rangle \in \mathcal{H}_j$ and $i \neq j$, then $\langle\, a\,|\, b\,\rangle = 0$. Taking it into account, we can write the decomposition (4) in the form:

$$\mathcal{H} = \mathcal{H}_0 \dot{+} \mathcal{H}_1 \dot{+} \mathcal{H}_2 \dot{+} \dots \tag{5}$$

In the case when the scalar product is positive-definite, the decomposition (5) allows to establish connection with the usual Fock construction [3]. So far as this is quite simple, we will not discuss this connection in more details.

4. Invariant quantization. In the section 2 quantization of a classical field was described in “internal” notions of the invariant Hamiltonian formalism. But for the choice of the subspaces Cr and De we have not suggested any concrete prescription: we have just given some necessary conditions for the choice of these subspaces. In practice it leads to the situation that for one classical field we can get too many quantizations.

Let us suppose now that in the space $Z^*_{\mathbb{C}}$ there is a linear symplectic representation of some group. Let us define then *invariant quantization* by the requirement that subspaces Cr and De are invariant with respect to the action of this group. In other words, Cr and De are reducing subspaces of this representation.

When we consider relativistic fields, as a main group of invariance we have Poincare group $\mathcal{P}$. A quantization, invariant with respect to the Poincare group we will call $\mathcal{P}$ -invariant, for shortness. In the paper [IV] we have already seen, how the Poincare group acts in the space $Z^*_{\mathbb{C}}$, and how its representations are reduced.

Quantization of relativistic fields we will consider a little later, but now we will consider quantization of the harmonic oscillator. Its symmetry group is just the one-parameter group of time shifts.

5. Quantization of harmonic oscillator. Consider harmonic oscillator. It is described by the Lagrangian:

$$L = \tfrac{1}{2}\dot{\varphi}^2 - \tfrac{m^2}{2}\varphi^2\ . \tag{6}$$

Here $\varphi(x)$ is a real function of one real argument (time).

The equation of the motion is:

$$(\partial^2 + m^2)\,\varphi = 0\ .$$

The invariant phase space Z is a two-dimensional real space. The symplectic structure on it is given by the formula:

$$\omega = \dot{\varphi}(t) \wedge \varphi(t)\ .$$

Here forms $\dot{\varphi}$ and φ are taken in the same arbitrary moment of time t .

So far as the Lagrangian (6) is invariant with respect to time shifts, in the space Z we have the action of the additive group $\mathbb{R}$. The field representation $Z^*_{\mathbb{C}}$ in this case, like in the case of relativistic fields, decomposes into the direct sum of positive- and negative-frequency subrepresentations:

$$Z^*_{\mathbb{C}} = Z^{*\,(+)}_{\mathbb{C}} \oplus Z^{*\,(-)}_{\mathbb{C}}\ .$$

As a reducing basis let us take the following two elements:

$$a = \tfrac{1}{\sqrt{2m}}\,(m\varphi(0) + i\dot{\varphi}(0))\ , \qquad a^* = \tfrac{1}{\sqrt{2m}}\,(m\varphi(0) - i\dot{\varphi}(0))\ .$$

Really, so far as we have decomposition $\varphi(t) = \frac{1}{\sqrt{2m}}(a\,e^{-imt} + a^* e^{+imt})$, we get that $a \in Z^{*\,(+)}_{\mathbb{C}}$, $a^* \in Z^{*\,(-)}_{\mathbb{C}}$.

Symplectic structure can be written by the forms a and a^* in the following way:

$$\omega = i\,a^* \wedge a\ .$$

The Poisson bracket of the corresponding linear functions is:

$$\{ a \, , \, a^* \} = -i \; .$$

So, in accordance with what we have said in the section 4, we have exactly two invariant quantizations: either $Cr = Z_{\mathbb{C}}^{*\,(-)}$ and $De = Z_{\mathbb{C}}^{*\,(+)}$, or $Cr = Z_{\mathbb{C}}^{*\,(+)}$ and $De = Z_{\mathbb{C}}^{*\,(-)}$. It is easy to see that the first quantization leads to positive-definite scalar product. In the case of second quantization scalar product turns out to be indefinite. And in the decomposition (5) on the subspaces with even number of particles the scalar product is positive-definite, and on subspaces with odd — negative.

6. Connection of invariant quantizations of field and of field oscillator. A field oscillator, as a finite-dimensional system, is more convenient for investigation by algebraic means. On the other hand, there is a close connection between quantization of a field and quantization of its oscillator.

Really, it is easy to see that the choice of $\mathbb{R} \times \mathcal{L}_{k^{(0)}}$ -invariant creation and destruction subspaces for oscillator and the choice of $\mathcal{P}$ -invariant creation and destruction subspaces for field are connected by the operation of induction. So, there is one-to-one correspondence between $\mathbb{R} \times \mathcal{L}_{k^{(0)}}$ -invariant quantizations of oscillator and $\mathcal{P}$ -invariant quantizations of field.

Furthermore, corresponding quantizations of oscillator and field have in the spaces of states scalar products either positive-definite or indefinite simultaneously. So, we get the convenient criterion of sign-definiteness of scalar product.

7. Quantization of scalar field. In the paper [IV] we have shown that with respect to the action of Poincare group the space of linear observables of scalar field $Z_{\mathbb{C}}^*$ decomposes into the direct sum of the two irreducible subspaces: $Z_{\mathbb{C}}^* = Z_{\mathbb{C}}^{*\,(+)} \oplus Z_{\mathbb{C}}^{*\,(-)}$. So, there are exactly two possibilities: either we accept $Cr = Z_{\mathbb{C}}^{*\,(-)}$ and $De = Z_{\mathbb{C}}^{*\,(+)}$, or we accept $Cr = Z_{\mathbb{C}}^{*\,(+)}$ and $De = Z_{\mathbb{C}}^{*\,(-)}$. According to the formulas for Poisson brackets of corresponding functions, obtained in the paper [IV], both decompositions satisfy all requirements of the section 2.

So, the scalar field allows exactly two $\mathcal{P}$ -invariant quantizations. The field oscillator in this case is just the usual one-dimensional real oscillator. Using results of the section 5 and the criterion from the section 6 we get that for the first quantization the scalar product in the space $\mathcal{H}$ turns out to be positive-definite, and for the second quantization - indefinite.

The first quantization is, in fact, conventional. Nevertheless, it should not be thought that quantization with indefinite metric is senseless, because it leads to "negative probabilities". In principle, we can not reject a usefulness of such a theory where scattering states of the field constitute a subspace where the scalar product is positive-definite.

8. Quantization of electromagnetic field. Consider now quantization of electromagnetic field. Under electromagnetic field we will imply here "non-physical" electromagnetic field [I].

As we have explained in the paper [IV], with respect to the action of the Poincare group $\mathcal{P}$ the space $Z_{\mathbb{C}}^*$ in this case, like in the case of the scalar field, decomposes into the direct sum of the two indecomposable subspaces. Therefore, like in the case of the scalar field, we have exactly two possibilities: either $Cr = Z_{\mathbb{C}}^{*\,(-)}$ and $De = Z_{\mathbb{C}}^{*\,(+)}$, or $Cr = Z_{\mathbb{C}}^{*\,(+)}$ and $De = Z_{\mathbb{C}}^{*\,(-)}$.

Using the criterion from the section 6, we easily see that both quantizations lead to indefinite metric. Later on we will consider only the first of the two quantizations, because this quantization has useful physical interpretation.

The indefiniteness of the scalar product leads, of course, to difficulties with probability interpretation.

In the paper [I] we have pointed out that scattering states of the classical electromagnetic field satisfy additional condition — the Lorentz condition. It is natural to suppose that in the quantum theory there is some analog of this condition. It is important to emphasize, that this condition must appear in the quantum theory not as a new postulate: it must be derivable from the dynamical laws, like we have done it for the classical field. Unfortunately, we do not have a satisfactory theory of interacting fields yet. Because of this reason, we will not give here any derivation[2]. The desired condition is:

$$- \, i \, k_\mu \, \hat{a}_\mu^{(+)}(k) \, | \, \mathrm{rad} \, \rangle = 0 \; . \tag{7}$$

[2]A formal derivation, of course, was present even in [11]. On the other hand, it is possible to appeal to Feynman rules, but this rules need grounding from the point of view of this paper.

So, a vector of state of radiated field satisfies this equality for any k.

The given condition looks the same like one of versions of this condition in the classical theory. Nevertheless, it should be noticed that it can not be written as $-i\,k_\mu\,\hat a_\mu^{(-)}(k)\,|\,\mathrm{rad}\,\rangle = 0$ or as $-i\,k_\mu\,\hat a_\mu(k)\,|\,\mathrm{rad}\,\rangle = 0$, because even the vacuum does not satisfy these conditions[3].

Now we will show that the condition (7) leads to positivity of scalar product.

9. Positivity of scalar product of electromagnetic field. [4] Consider the field oscillator of electromagnetic field. Let us write here, for shortness, a_μ instead of $\hat a_\mu(+1)$ and a^*_μ instead of $\hat a_\mu(-1)$. These operators satisfy relations:

$$[\,a^*_\mu\,,\,a_\nu\,] = g_{\mu\nu}, \qquad [\,a^*_\mu\,,\,a^*_\nu\,] = 0, \qquad [\,a_\mu\,,\,a_\nu\,] = 0\ .$$

For the quantization under consideration we also have:

$$a_\mu\,|\,0\,\rangle = 0\ .$$

The additional condition takes the form:

$$k^{(0)}_\mu a_\mu\,|\,\mathrm{rad}\,\rangle = 0\ , \tag{8}$$

here $k^{(0)}_\mu$ is a fixed vector on the light cone. The subspace of the vectors of states satisfying this condition we will denote $\mathcal{H}^{\mathrm{rad}}$.

So far as the operator $k^{(0)}_\mu a_\mu$ has a definite grade (its grade is equal to -1), the graduation of the space of states $\mathcal{H}$ is transferred to the space $\mathcal{H}^{\mathrm{rad}}$:

$$\mathcal{H}^{\mathrm{rad}} = \mathcal{H}^{\mathrm{rad}}_0 \dot{+} \mathcal{H}^{\mathrm{rad}}_1 \dot{+} \mathcal{H}^{\mathrm{rad}}_2 \dot{+} \dots \tag{9}$$

In other words, the subspace, defined by the additional condition (8), also decomposes into the orthogonal sum of states with definite number of particles.

So far as the sum (9) is orthogonal, it is enough to prove the non-negativity of the scalar product for each of the subspaces $\mathcal{H}^{\mathrm{rad}}_n$.

An arbitrary state vector $|\,n\,\rangle$ from the subspace $\mathcal{H}^{\mathrm{rad}}_n$ can be represented in the form:

$$|\,n\,\rangle = T_{\mu\nu\dots\rho}\,a^*_\mu\,a^*_\nu \dots a^*_\rho\,|\,0\,\rangle\ ,$$

here $T_{\mu\nu\dots\rho}$ is an n-valent tensor. This tensor can be supposed to be symmetrical, without loss of generality.

The condition (8) for the vector $|\,n\,\rangle$ turns out to be an additional condition for the tensor $T_{\mu\nu\dots\rho}$:

$$k^{(0)}_\mu T_{\mu\nu\dots\rho} = 0\ . \tag{10}$$

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[3] In coordinate representation the condition $-i\,k_\mu\,\hat a_\mu(k)\,|\,\mathrm{rad}\,\rangle = 0$ looks as $\partial_\mu \hat A_\mu(x)\,|\,\mathrm{rad}\,\rangle = 0$. In literature even up to now there is no unity of opinions on whether or not it is possible to formulate the additional condition in this way. In this way, on the grounding of analogy with the classical field, it was formulated even by Fermi. It was pointed out in the papers [12, 13] that such an additional condition leads to difficulties with normalizability of states in the quantum case. In the paper [14] it was substituted with $\partial_\mu\,\hat A^{(+)}_\mu(x)\,|\,\mathrm{rad}\,\rangle = 0$. Later in literature some authors used condition $\partial_\mu\,\hat A_\mu(x)\,|\,\mathrm{rad}\,\rangle = 0$ [15, 16, 17]. Other [18] pointed that such a condition can not be applied, because there appears some contradiction with commutation relations. The other authors [19] claimed that such a condition can be used, but special care is needed, because allowed vectors of states turn out to be not-normalizable. These differences of opinion have quite deep reasons:

1. No exact meaning was put into the term "quantization". In fact, quantized electromagnetic field was studied by formal manipulations with algebraic symbols, but it was not given any constructive definition of this object. In such a situation the additional condition in any form is not free from criticism.
2. There is no satisfactory formulation of the theory of interacting fields (even in the frame of perturbation theory). And the Feynman rules do not have any clear connection with the operatoral formalism, and operatoral formalism turns out to be separated from practical calculations (from scattering theory, first of all).

With regards to the first remark, I believe, the relation $\partial_\mu\,\hat A_\mu(x)\,|\,\mathrm{rad}\,\rangle = 0$ does not contradict directly to the commutation relations. It is possible to say only that the construction of quantization presented in this paper is badly co-ordinated with such a condition.

The second remark is, of course, more important. When we considered the dynamics of the classical field, it was pointed out that the additional condition for scattering states automatically follows from dynamics, and it is not an arbitrary postulate. We have also clearly shown the role of the "non-physical" degrees of freedom. And though in this paper we still do not formulate a theory of quantum interacting fields, nevertheless, arguments based on analogy with the classical theory give us reasons to think that construction of theory with condition $\partial_\mu\,A_\mu(x)\,|\,\mathrm{rad}\,\rangle = 0$ is not a realistic task.

[4] The content of this section does not have any deep connection neither with constructivity of quantization, nor with algebraic, nor with topological questions discussed in the present papers. The only reason why we discuss here this old (and very simple) question is that in all sources that I know an erroneous proof is given (for some reason people think that positivity of a quadratic form on vectors of some basis necessarily leads to positivity of the form in general).

The scalar product for the vector $|\,n\,\rangle$ multiplied by itself is:

$$\langle\, n\,|\, n\,\rangle = (-1)^n\, n! \cdot T^*_{\mu\nu\ldots\rho}\, T_{\mu\nu\ldots\rho} \ . \tag{11}$$

The value $(-1)^n\, n! \cdot T^*_{\mu\nu\ldots\rho}\, T_{\mu\nu\ldots\rho}$ is a sum of positive numbers, some of them are included in the sum with the sign “plus”, and some of them with the sign “minus”. This sign is defined by the relativistic summation rule by repeated indexes and by the factor $(-1)^n$. Let us show that this sum is non-negative under condition (10).

So far as the construction of the space of states is invariant with respect to the choice of the vector $k^{(0)}_\mu$, this vector can be supposed to be equal:

$$k^{(0)}_\mu = \left(\ k_0 \ \middle|\ 0 \quad 0 \quad k_0\ \right)_\mu \ . \tag{12}$$

Consider now in the sum (11) all addends of the type $(-1)^n\, n! \cdot T^*_{0\nu\ldots\rho}\, T_{0\nu\ldots\rho}$. Because of (10) and (12), they are completely neutralized by the addends of the type $(-1)^n\, n! \cdot T^*_{3\nu\ldots\rho}\, T_{3\nu\ldots\rho}$.

Among remaining addends let us consider addends of the type $(-1)^n\, n! \cdot T^*_{\mu 0\ldots\rho}\, T_{\mu 0\ldots\rho}$. They are completely neutralized with the remaining addends of the type $(-1)^n\, n! \cdot T^*_{\mu 3\ldots\rho}\, T_{\mu 3\ldots\rho}$.

Et cetera. Finally, in the sum (11) there remain only addends that have indexes 1 or 2 . All these addends are included in the sum (11) with the sign “plus”.

10. Topology and completeness of invariant phase space of electromagnetic field. When we studied the invariant Hamiltonian formalism, at the beginning we included into the invariant phase space Z only solutions of the equations of the motion that are smooth in coordinate representation and finite in space direction. But the question about what topology should be really introduced in the space Z we have not discussed at all. In fact, we have not given even an exact definition of the space Z^* ; we have not clarified, how exactly the functional spaces Z and Z^* turn out to be isomorphic; an exact definition of the Poisson brackets on Z^* was not given also. In this section we will show, that in the space Z we can introduce such a topology (and to complete it with respect to this topology), that it becomes very similar to the Hilbert space. And all necessary for invariant Hamiltonian formalism properties of this topology will be satisfied.

A symplectic structure itself does not define a topology. But between elements of the classical phase space Z and one-particle states of a quantized field there exist one-to-one correspondence:

$$\underline{c} \ \longleftrightarrow\ (I^{-1}\underline{c})\widehat{}\ |\,0\,\rangle \ . \tag{13}$$

Consider now the scalar field as an example. Among its invariant quantizations there is a quantization with positive-definite scalar product. The correspondence (13), firstly, introduces a complex structure in the space Z , and secondly, transfers there the scalar product from the one-particle quantum space $\mathcal{H}_1$. So, Z has natural structure of complex Hilbert space[5].

The symplectic structure turns out to be continuous with respect to the pair of its arguments with respect to the obtained topology; the symplectic structure defines the isomorphism of the spaces Z and Z^* ; the Poisson brackets are correctly defined on the whole Z^* .

Let us represent now the scalar field in the Fourier representation in the form:

$$\widetilde{\varphi}(k) = 2\pi\, \delta(k^2 - m^2)\cdot a(k) \ .$$

The correspondence (13) can be written more manifestly as:

$$\underline{c} \ \longleftrightarrow\ \int d\mu^+_m \cdot i\, a(k)^{\underline{c}} \cdot \hat{a}^*(k)\ |\,0\,\rangle \ , \qquad \text{here } \ d\mu^+_m = \frac{d^4k}{(2\pi)^4}\cdot 2\pi\, \delta(k^2-m^2)\cdot\theta(k) \ .$$

And the scalar product in the space Z takes the form:

$$\langle\, \underline{c}\, ,\, \underline{d}\,\rangle = \int d\mu^+_m \cdot a^*(k)^{\underline{c}} \cdot a(k)^{\underline{d}} \ . \tag{14}$$

[5]The obtained topology is the most natural for quantization. Bohr and Rosenfeld [20] noticed that for consideration of the quantized electromagnetic field it is necessary to perform an averaging of the field in a small space-time volume: the symbol $\hat{A}_\mu(x)$ itself is not a physical value. Our consideration shows that this statement is in the same way true for the classical field also, if its phase space is provided with the “proper” topology.

The one-to-one transference of this scheme to the case of electromagnetic field appears to be impossible, because, as we have seen, among $\mathcal{P}$ -invariant quantizations of electromagnetic field there are no quantizations with positive-definite scalar product.

But here is another opportunity. Let us represent electromagnetic field in the Fourier representation as

$$\widetilde{A}_\mu(k) = 2\pi\,\delta(k^2)\cdot a_\mu(k)\ .$$

By analogy with the formula (14) let us introduce in the classical phase space Z of electromagnetic field a scalar product by the formula:

$$\langle\, \underline{c}\,,\, \underline{d}\,\rangle = \int d\mu_m^+ \cdot M_{\nu\rho}\cdot a_\nu^*(k)^{\,\underline{c}}\cdot a_\rho(k)^{\,\underline{d}}\ . \tag{15}$$

Here $M_{\nu\rho}$ is an arbitrary positive-definite Hermitian matrix. In the capacity of such a matrix we can take, for example, $M_{\nu\rho} = \mathrm{diag}\,(+1,+1,+1,+1)_{\nu\rho}$.

The scalar product (15) is positive-definite. Therefore it defines some topology in the space Z . The important is the fact that this topology does not depend on the concrete choice of matrix $M_{\nu\rho}$.

From this it follows that the introduced topology is relativistic-invariant.

With respect to the introduced topology the symplectic structure turns out to be continuous with respect to the pair of its arguments; the symplectic structure defines an isomorphism of the spaces Z and Z^* ; the Poisson brackets turn out to be correctly defined on the whole Z^* .

So, the invariant phase space of electromagnetic field has natural structure of linear complex topological space, and the topology there is defined by a set of equivalent (in the sense of topology) scalar products. Such spaces we will call *spaces of Hilbert type.*

11. Tensor product of spaces of Hilbert type. Now we will show that tensor product of spaces of Hilbert type is a space of Hilbert type.

Consider two such spaces X and Y . Their tensor product (more exactly, algebraic tensor product) is defined as follows.

Consider all formal products of the type $x \diamond y$, where $x \in X$ and $y \in Y$. We will call such products pairs. Let us suppose that pairs can be formally multiplied by complex numbers forming expressions of the type: $\lambda\cdot x\diamond y$. And consider all formal sums of the type:

$$\lambda_1\cdot x_1\diamond y_1 + \lambda_2\cdot x_2\diamond y_2 + \ldots + \lambda_n\cdot x_n\diamond y_n\ .$$

Let us consider two such sums to be equivalent, if they have the same sums of factors for each pair. So, we come to the complex vector space that we denote as $X\diamond Y$ (this is a *free* $\mathbb{C}$ *-module generated by the Cartesian product* $X\times Y$).

In the constructed space consider elements of the type:

$$1\cdot(\lambda x)\diamond y - \lambda\cdot x\diamond y\ ,$$

$$1\cdot x\diamond(\lambda y) - \lambda\cdot x\diamond y\ ,$$

$$1\cdot(x_1+x_2)\diamond y - 1\cdot x_1\diamond y - 1\cdot x_2\diamond y\ ,$$

$$1\cdot x\diamond(y_1+y_2) - 1\cdot x\diamond y_1 - 1\cdot x\diamond y_2\ .$$

The linear shell of these elements we denote as $X\circ Y$.

Factorizing $X\diamond Y$ with respect to $X\circ Y$ we come to the tensor product:

$$X\otimes Y = (X\diamond Y)/(X\circ Y)\ .$$

For shortness, let us introduce the notation $x\otimes y$. Namely, let us suppose that under the defined factorization an element $1\cdot x\diamond y$ of the space $X\diamond Y$ transfers into the element $x\otimes y$ of the space $X\otimes Y$.

If X and Y are usual Hilbert spaces, then, as it is known, their tensor product $X\otimes Y$ has natural structure of Hilbert space. The scalar product in $X\otimes Y$ in this case at the beginning is defined for pairs by the formula:

$$\langle\, x_1\otimes y_1\,,\, x_2\otimes y_2\,\rangle_{X\otimes Y} = \langle\, x_1\,,\, x_2\,\rangle_X\cdot\langle\, y_1\,,\, y_2\,\rangle_Y\ , \tag{16}$$

and for other elements it is spread by the requirement of linearity with respect to the second argument and anti-linearity with respect to the first. Performing completion of $X \otimes Y$ with respect to the given scalar product we come to Hilbert space.

In the case of spaces of Hilbert type each of the spaces X and Y has many equivalent scalar products. The space $X \otimes Y$, in accordance with the given scheme, gets many scalar products. Let us show that all these scalar products are equivalent.

So, suppose that in one of the two spaces, for example in X, we change the scalar product $\langle \cdot , \cdot \rangle_X$ to equivalent scalar product $(\cdot , \cdot)_X$. And the scalar product $\langle \cdot , \cdot \rangle_{X \otimes Y}$ in the space $X \otimes Y$ changes to $(\cdot , \cdot)_{X \otimes Y}$.

Consider some element of the space $X \otimes Y$:

$$z = x_1 \otimes y_1 + x_2 \otimes y_2 + \ldots + x_n \otimes y_n \, .$$

It is easy to see that this element can be represented in such a form that in this sum all $y_1, y_2, \ldots, y_n$ are orthogonal to each other with respect to the scalar product in Y. Then the scalar products of this element with itself take especially simple form:

$$\langle z \, , \, z \rangle_{X \otimes Y} = \langle x_1 \, , \, x_1 \rangle_X \cdot \langle y_1 \, , \, y_1 \rangle_Y + \langle x_2 \, , \, x_2 \rangle_X \cdot \langle y_2 \, , \, y_2 \rangle_Y + \ldots + \langle x_n \, , \, x_n \rangle_X \cdot \langle y_n \, , \, y_n \rangle_Y \, , \tag{17}$$

$$(z \, , \, z)_{X \otimes Y} = (x_1 \, , \, x_1)_X \cdot \langle y_1 \, , \, y_1 \rangle_Y + (x_2 \, , \, x_2)_X \cdot \langle y_2 \, , \, y_2 \rangle_Y + \ldots + (x_n \, , \, x_n)_X \cdot \langle y_n \, , \, y_n \rangle_Y \, . \tag{18}$$

A convenient necessary and sufficient criterion of equivalence of scalar products $\langle \cdot , \cdot \rangle_{X \otimes Y}$ and $(\cdot , \cdot)_{X \otimes Y}$ is the following: there exists such $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, that for any $z \in X \otimes Y$ we have the following conditions:

$$\varepsilon \cdot (z \, , \, z)_{X \otimes Y} < \langle z \, , \, z \rangle_{X \otimes Y} \, , \qquad \varepsilon \cdot \langle z \, , \, z \rangle_{X \otimes Y} < (z \, , \, z)_{X \otimes Y} \, .$$

But if such ε exists for the scalar products $\langle \cdot , \cdot \rangle_X$ and $(\cdot , \cdot)_X$, then in accordance with formulas (17) and (18), it is appropriate also for the scalar products $\langle \cdot , \cdot \rangle_{X \otimes Y}$ and $(\cdot , \cdot)_{X \otimes Y}$.

So, all scalar products in $X \otimes Y$ are equivalent. Performing completion of $X \otimes Y$ with respect to the topology defined by these scalar products we come to the space of Hilbert type.

12. Topology of space of states of quantized electromagnetic field. As we have seen, the space of states of quantized electromagnetic field $\mathcal{H}$ decomposes into orthogonal sum of subspaces with definite number of particles:

$$\mathcal{H} = \mathcal{H}_0 \dot{+} \mathcal{H}_1 \dot{+} \mathcal{H}_2 \dot{+} \ldots \tag{19}$$

The space $\mathcal{H}_0$ is one-dimensional. So, the question about its topology does not arise.

The subspace $\mathcal{H}_1$, as it was shown in the section 10, can be in fact identified with the phase space of the classical field Z. It is the space of Hilbert type. So, the question about its topology is also solved.

Consider now the tensor product of the space Z with itself: $Z \otimes Z$. There we have naturally defined action of the group of transpositions of two elements. The simplest elements are transformed under action of this group as $\underline{a} \otimes \underline{b} \to \underline{b} \otimes \underline{a}$, and to other elements this action is spread by linearity.

Let us choose among the scalar products in the space $Z \otimes Z$ only those invariant with respect to the action of the group under consideration. In practice it is convenient just to restrict ourself with the products for which in the right part of the formula (16) we have product of the same scalar products.

Then the group of transpositions acts unitarily[6] in $Z \otimes Z$. And $Z \otimes Z$ decomposes into the orthogonal sum of two invariant subspaces: symmetric and anti-symmetric.

Each of these two subspaces inherits topology from $Z \otimes Z$.

It is easy to see that the two-particle quantum space $\mathcal{H}_2$ can be naturally identified with the symmetric subspace in $Z \otimes Z$ and therefore it gets corresponding topology.

Et cetera. n-particle subspace $\mathcal{H}_n$ is identified with fully symmetric subspace of n-th tensor power of Z. And inherits topology from there.

[6] Under unitary transformation of a space of Hilbert type we imply an automorphism of this space, i. e. one-to-one transformation on itself preserving linear structure and each of scalar products.

So, each of the subspaces in the orthogonal sum (19) is a space of Hilbert type. From practical point of view it is very convenient, because for practical applications it is much easier to define the described spaces using bases (and not as factor-spaces of free modules).

Consider now the space $\mathcal{H}$ as a whole. Using topologies introduced in its subspaces we can introduce different topologies in the whole space. For example, convergence of a directed set to zero can be understood as independent convergence to zero of all projections of this set. Such a topology seems to be the most natural in this case.

On the other hand, this requirement can be strengthened if we require additionally, for example, that starting from some moment only finite number of projections of the directed set differ from zero.

So, there are many natural topologies in $\mathcal{H}$. And the space $\mathcal{H}$ is not a space of Hilbert type.

In this connection I want to pay attention to the following. In the paper of Gupta [14] it was suggested to quantize electromagnetic field in usual Hilbert space with positive-definite metric. In fact, it is possible to do so considering quantizations invariant with respect to a more narrow group than the Poincare group (namely, with respect to the subgroup of the Poincare group that leaves the direction of the time axis unchanged). The subspaces with fixed number of particles turn out to be the same, in fact, that we constructed above. But the whole space $\mathcal{H}$ acquires such a topology that is not relativistic-invariant. And it turns out that some states of the field belong to the Hilbert space in one frame of reference and do not belong in other. So, the old Gupta formalism is not relativistic-invariant, even implicitly.

In his later papers Gupta tried to refuse from forced introduction of a sign-definite metric. But his last paper on this topic [22] clearly showed that he still had no apparently relativistic-invariant construction of quantized electromagnetic field (to say nothing of the problems with functional analysis).

13. About origin of anti-unitary transformations. In accordance with the introduced procedure of quantization, linear transformations of observables of a classical field that preserve Poisson brackets and the subspaces Cr and De generate unitary transformations of states of the quantized field. But it is known that in quantum theory an important role is played also by anti-unitary symmetry transformations. Consider their origin on the example of the operation of time reversal — T .

Suppose that the Lagrangian of a classical field is T -invariant (as examples we can use scalar and electromagnetic fields). Then the action turns out to be T -invariant also. Therefore, field functions satisfying the stationary action principle under the operation of time reversal transfer into functions that also satisfy the stationary action principle.

So, from the point of view of invariant Hamiltonian formalism, the operation of time reversal is a one-to-one transformation of the phase space Z on itself. And how does the symplectic structure change? It is obvious from the variational definition of symplectic structure that it changes sign. It seems to be natural to call such transformations, that change sign of symplectic structure, *anti-symplectic.*

Consider now conjugate action of the operation of time reversal on elements of the conjugate space $Z^*_{\mathbb{C}}$:

$$a \xrightarrow{T} a_T \ , \qquad a, a_T \in Z^*_{\mathbb{C}} \ .$$

The Poisson bracket changes sign under such a transformation:

$$\{\, a_T \,,\, b_T \,\} = -\,\{\, a \,,\, b \,\} \ .$$

So, the operation of time reversal generates the automorphism of the Lie algebra $\mathcal{C} \oplus Z^*_{\mathbb{C}}$.

In accordance with the procedure of quantization described in this paper, it is natural to wish to construct corresponding automorphism of the algebra of operators $\mathcal{O}$ for the quantized field. But it is impossible to do it. The reason is that for construction of the algebra of operators we used not only the structure of the Lie algebra of the set $\mathcal{C} \oplus Z^*_{\mathbb{C}}$ but also the special correspondence between the subspace $\mathcal{C}$ and the set of scalars $\mathbb{C}$ (the relation $\hat{1} = ()^{\frown}$). It means that for bringing of the operation of time reversal to the quantum case we need some additional conventions, besides operation of quantization that we have already introduced.

Let us agree that words (i. e. elements of the semi-group $\mathcal{W}$) under time reversal are transformed in accordance with the following formula:

$$\hat{a}\,\hat{b}\ldots\hat{c} \xrightarrow{T} \hat{c}_T\ldots\hat{b}_T\,\hat{a}_T \ , \qquad \hat{a},\hat{b},\ldots,\hat{c},\hat{a}_T,\hat{b}_T,\ldots,\hat{c}_T \in \mathcal{A} \ ,$$

i. e. every letter is substituted by the corresponding, and after that letters are written in the reverse order. It is obvious that time reversal acts on a product of two words in the following way:

$$\hat{p}\,\hat{q} \xrightarrow{T} \hat{q}_T\,\hat{p}_T \ , \qquad \hat{p},\hat{q},\hat{p}_T,\hat{q}_T \in \mathcal{W} \ .$$

A one-to-one transformation of a semi-group on itself that satisfies such a property can be called an *anti-automorphism.*

The constructed anti-automorphism of the semi-group of words $\mathcal{W}$ by linearity spreads to anti-automorphism of the algebra of phrases $\mathcal{P}$. Furthermore, it is easy to check that this anti-automorphism of the algebra of phrases generates an anti-automorphism of the algebra of operators $\mathcal{O}$.

Suppose now that under time reversal creation subspace Cr and destruction De transfer to one another. This condition is usually satisfied because these subspaces are negative- and positive-frequency, correspondingly. The left ideal spanned on $De^{\frown}$ under time reversal transfers into the right ideal spanned on $Cr^{\frown}$. The constructed by factorization left module transfers into the corresponding right module.

So, the operation of time reversal defines the one-to-one linear correspondence of ket- and bra-vectors[7].

As it can be easily seen, the scalar product is preserved under time reversal:

$$\langle\, x \,|\, y \,\rangle = \langle\, y_T \,|\, x_T \,\rangle \ . \tag{20}$$

So far as there is a one-to-one anti-linear correspondence between ket- and bra-vectors, we could make all our discussion inside one space, for example, the space of ket-vectors. Then the operation of time reversal could be considered as one-to-one anti-linear transformation of this space. The relation (20) shows that under this transformation the scalar product changes to complex-conjugate. A transformation satisfying such properties is called *anti-unitary.*

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