On quantization of electromagnetic field.
V. Vector representation of little Lorentz group for light-like momentum.

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Abstract

Using elementary geometric methods we prove the isomorphism of the little Lorentz group for light-like momentum and the group of motions of a Euclidian plane. In accordance with Jordan-Hölder-Noether theorem we perform "reduction" of the real and complex vector representations of this group. We also prove indecomposability of these representations.

1. Preliminary remarks. There are quite many books devoted to the theory of the Lorentz group $\mathfrak{L}$ and the theory of representations of this group. Among them there are monographs specially devoted to this topic; for example, [1, 2, 3]. It is mentioned in all them that the little Lorentz group $\mathfrak{L}_k$ for a light-like momentum[1] k is isomorphic to the group of motions of a Euclidian plane $E(2)$. The proof that is usually provided is usually based on an analytic investigation of the group $SL(2,\mathbb{C})$ which is the universal covering group of the Lorentz group.

Such an approach seems to be optimal if we want to study then arbitrary (including two-valued) representations of the group $\mathfrak{L}_k$. At the same time it turns out to be very difficult to realize the geometric sense of the isomorphism of $\mathfrak{L}_k$ and $E(2)$ [2].

Let us specify here what is usually implied when we talk about "geometric sense". From the point of view of physical applications the Lorentz group is most naturally defined by its vector representation[3]. And the little group $\mathfrak{L}_k$ is defined just as the stationary subgroup of the vector k . The group $E(2)$ in its turn is defined as the group of motions of a Euclidean plane. Taking into account these definitions of the groups $\mathfrak{L}_k$ and $E(2)$ and using purely geometric constructions we will find some two-dimensional plane, possessing natural Euclidean structure, where the group $\mathfrak{L}_k$ acts as the group of motions.

Another question that we will study here (and which is the main motivation for publishing of this paper) is the "reduction" of vector representation of the group $\mathfrak{L}_k$, i. e. we will expose what irreducible representations it consists of. From the point of view of the general representation theory, this case is very particular and does not contain any specific difficulties. Nevertheless, taking into account its great practical importance, it seems to be useful to perform its investigation with using constructions that are closest to our geometric intuition.

Real vector representation

2. Notations and terminology. *The Minkowski space* M is a four-dimensional real linear space $\mathbb{R}^4$, where we have real symmetrical bilinear form $g(\cdot,\cdot)$ with the signature $(+,-,-,-)$. This form is called *scalar product.* For short, instead of writing $g(a,b)$, we will write just ab or $a\cdot b$.

Consider now the group of all linear transformations of the Minkowski space M preserving the scalar product. The connected component of this group containing neutral element is a group also. This latter group we will call *the Lorentz group* and we will denote it $\mathfrak{L}$.

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[1] The word "momentum" is used because we keep in mind applications to the theory of zero-mass fields; in particular electromagnetic. While we perform purely mathematical investigation, we can suppose that we talk about any isotropic vector k : $k^2=0$.

[2] In particular, Ryder [4] noted about it that the physical meaning of this result is unclear.

[3] Such a representation is sometimes called "fundamental".

In this paper we will consider only the vector representation of the Lorentz group, i. e. the representation in the Minkowski space. This representation will be denoted as T .

The little group of a vector k is the stationary subgroup of the vector k , i. e. the subgroup of the Lorentz group leaving the vector k unchanged. It will be denoted as $\mathfrak{L}_k$.

A vector k is called *time-like*, if $k^2 > 0$; *light-like* or *isotropic*, if $k^2 = 0$; and *space-like*, if $k^2 < 0$.

Consider also the group of all transformations of a Euclidean plane preserving distances. The connected component of this group containing neutral element is a group also. We will call it *the group of motions of a Euclidean plane* and denote it $E(2)$.

3. Reducing subspaces. Following the Jordan-Hölder-Noether theorem let us immediately guess a composition series of the representation T , i. e. such a monotonic set of reducing subspaces of the representation T that corresponding successive factor-representations are irreducible:

$$\{0\} = M^0 \subset M^{\parallel} \subset M^{\perp} \subset M^4 = M \ . \tag{1}$$

Here M^0 is the zero subspace in M ; $M^{\parallel}$ is the subspace containing all vectors parallel to k ; $M^{\perp}$ is the subspace containing all vectors orthogonal to k ; M^4 is another notation for the Minkowski space.

Each of these subspaces is obviously invariant with respect to the action of the group $\mathfrak{L}_k$. The subrepresentations in the subspaces $M^{\parallel}$ and $M^{\perp}$ we denote $T^{\parallel}$ and $T^{\perp}$, correspondingly.

For the set (1) we have the corresponding sequence of the three successive factor-representations in the factor-spaces $M^{\parallel}/M^0 = M^{\parallel}$, $M^{\perp}/M^{\parallel} = M^{\perp/\parallel}$ and $M^4/M^{\perp} = M^{4/\perp}$. The first of these factor-representations is a subrepresentation. We have already denoted it as $T^{\parallel}$. The second and the third factor-representations we denote $T^{\perp/\parallel}$ and $T^{4/\perp}$, correspondingly.

The irreducibility of the factor-representations $T^{\parallel}$ and $T^{4/\perp}$ is undoubted because they are one-dimensional. As regards the factor-representation $T^{\perp/\parallel}$, its irreducibility (real) will be proved in the section 6.

In the sections 7 and 8 we will prove two lemmas that have as a consequence that besides the subspaces (1) the representation T does not have any other reducing subspaces. So, the composition series (1) is the only one. Therefore in this case the statement of the Jordan-Hölder-Noether theorem that all composition series are of the same length and corresponding factor-representations of these series are equivalent turns out to be trivial.

4. Homomorphism $\mathfrak{L}_k$ into $E(2)$. Consider now in the Minkowski space a three-dimensional hyperplane N defined by the equation:

$$k \cdot x = const \neq 0 \ . \tag{2}$$

So far as the group $\mathfrak{L}_k$ leaves the vector k unchanged and preserves the scalar product, the hyperplane N is invariant with respect to the action of the group $\mathfrak{L}_k$.

Let us introduce now an equivalence relation $\sim$ for vectors from N :

$$a \sim b \iff a - b \in M^{\parallel} \ , \tag{3}$$

In other words, two vectors are equivalent if their ends belong to a line parallel to k . Because of (1) and (2) every such a line completely lies in the hyperplane N . Making factorization of N with respect to the equivalence relation (3) we get some two-dimensional plane[4] $\widetilde{N}$. We will denote points from $\widetilde{N}$ and lines corresponding to them on N by the same symbols: $\widetilde{a}$, $\widetilde{c}$ etc.

The plane $\widetilde{N}$ has natural Euclidean structure. Really, let us take two arbitrary points $\widetilde{a}$ and $\widetilde{c}$ on $\widetilde{N}$. Both of them are equivalence classes of points on N . Let us take a one arbitrary representative for each of the two classes: a and c , correspondingly. Let us calculate the value

$$\rho\,(a, c) = \sqrt{(a - c)^2} \ . \tag{4}$$

This value does not depend on the choice of the representatives a and c . Really, let us substitute, for example, a with $a + \alpha k$. Taking into account (2), we have:

$$\rho\,(a + \alpha k, c) = \sqrt{(a - c + \alpha k)^2} = \sqrt{(a - c)^2 + 2\alpha(ka - kc) + \alpha^2 k^2} = \sqrt{(a - c)^2} = \rho\,(a, c) \ .$$

[4]The plane $\widetilde{N}$ is not *imbedded* into the Minkowski space.

What is more, $(a-c)^2$ is always greater than zero and the function $\rho\,(\,\cdot\,,\,\cdot\,)$ have all properties of Euclidean metric, because the plane $\widetilde{N}$ can be identified with any of space-like two-dimensional planes lying in N .

Note now that the equivalence relation (3) and metric $\rho\,(\,\cdot\,,\,\cdot\,)$ are invariant with respect to the action of the little group $\mathfrak{L}_k$. So, we have constructed a homomorphous mapping of the group $\mathfrak{L}_k$ into the group of motions $E(2)$ of the Euclidean plane $\widetilde{N}$.

5. Inverse mapping. Later on we will repeatedly use the following two statements that are true in both the real and the complex cases.

S t a t e m e n t. *If a linear transformation changes some basis* $\{n^i\}_{i=1..4}$ *so that scalar products of all elements of the basis remain unchanged (including squares of the elements of the basis), then this transformation preserves the scalar product.*

S t a t e m e n t. *If a linear transformation changes some basis* $\{n^i\}_{i=1..4}$ *so that all values* $(n^i-n^j)^2$ *and* $(n^i)^2$ *remain unchanged, then this transformation preserves the scalar product.*

The first statement follows from the fact that every vector can be decomposed into a linear combination of the elements of basis. The second statement follows from the first, if we take into account the formula:

$$(n^i-n^j)^2=(n^i)^2+(n^j)^2-2n^in^j\ .$$

Let us fix now an arbitrary motion of the plane $\widetilde{N}$. A transformation from the little group $\mathfrak{L}_k$, that transforms to this motion on $\widetilde{N}$ under the homomorphism constructed in the section 4, we will call *desired*. Let us show that the desired transformation from $\mathfrak{L}_k$ exists and it is unique. So, we will prove that the homomorphism under consideration is really an isomorphism.

For that consider a family of hypersurfaces defined by the equations like

$$x^2=const\ . \tag{5}$$

The intersection of any of them with the hyperplane N is a paraboloid. Every such a paraboloid is invariant with respect to the desired transformation. And every line representing a point from $\widetilde{N}$ intersects such a paraboloid exactly in one point, and for every point on the paraboloid there is exactly one such a line that contains this point. Therefore we have defined the action of the desired transformation on every paraboloid.

Let us take now on one of these paraboloids four points that do not lie in any two-dimensional plane. The four vectors $\{n^i\}_{i=1..4}$ that have their ends at these four points form a basis in M . The transformation on the paraboloid that we have just defined preserves values $(n^i-n^j)^2$, because each of these values is a square of the corresponding distance in $\widetilde{N}$. The values $(n^i)^2$ are also preserved because they are constant on the paraboloid. Let us linearly continue the constructed transformation of the vectors from the basis to the whole space M . It follows from the second statement formulated in the beginning of this section that we will get a transformation preserving the scalar product in M . And according to our construction, this transformation preserves the plane (2), and therefore it leaves the vector k unchanged. So far as the initial motion of the plane $\widetilde{N}$ was supposed to belong to the connected group $E(2)$, the constructed transformation in M also belongs to the connected group $\mathfrak{L}_k$.

The constructed transformation is the only one that can pretend to be the desired. On the other hand, the constructed transformation from $\mathfrak{L}_k$ under the homomorphism from the section 4 transforms into some motion on $\widetilde{N}$. This motion, in accordance with the construction, acts on the four points[5] $\widetilde{n}^i\in\widetilde{N}$ in exactly the same way as the initial motion in $\widetilde{N}$. So far as a motion in a Euclidean plane is uniquely defined even by its action on two not-coinciding points, the constructed transformation from $\mathfrak{L}_k$ is the desired. So, we have proved the

T h e o r e m. *The groups* $\mathfrak{L}_k$ *and* $E(2)$ *are isomorphic.*

6. Factor-representation $T^{\perp/\parallel}$ **.** Similarly to the constructed in the section 4 factor-plane, the factor-space $M^{\perp/\parallel}$ has natural Euclidean structure[6], this structure is defined by the same formula (4). So, the little group $\mathfrak{L}_k$ acts in $M^{\perp/\parallel}$ as the group of rotations of a Euclidean plane over a fixed point. Such a group is denoted as $SO(2)$.

Obviously, in a Euclidean plane there is no such a direction that would be invariant under rotations. From this we get the

T h e o r e m. *The factor-representation* $T^{\perp/\parallel}$ *is irreducible (in the real sense).*

[5]The lines containing the ends of the vectors of basis $\{n^i\}_{i=1..4}$

[6]And what is more, $M^{\perp/\parallel}$ is linear, and there is the corresponding scalar product in it: $\widetilde{a}\cdot\widetilde{c}=ac$.

7. Indecomposability of representation T . Let us suppose that the representation T is decomposable, i. e. the space M can be represented as a direct sum $M = M' \oplus M''$ of two non-trivial reducing spaces M' and M'' . One of these spaces, for example M' , necessarily contains some vector that does not belong to $M^\perp$. If we properly choose the constant in the equation (2), the end of this vector will lie in the plane N . It was shown in the section 5 that by transformations from $\mathfrak{L}_k$ this vector can be transformed into any other vector, if the end of the latter lies on the same paraboloid. In particular, it can be transformed into four vectors $\{n^i\}_{i=1..4}$ forming a basis in M . Therefore, M' must coincide with the whole M . So, we have proved the

L e m m a. *Every reducing subspace of the representation T either coincides with the whole M or lies in $M^\perp$.*

From this lemma we get the

T h e o r e m. *The representation T is indecomposable.*

8. Indecomposability of representation $T^\perp$. Consider a cylinder defined by equations:

$$k \cdot x = 0 \ , \qquad x^2 = const \neq 0 \ . \tag{6}$$

Let us prove that $\mathfrak{L}_k$ acts transitively on it, i. e. every point on the cylinder can be transformed to any other. Let us denote the vector that have the end in the initial point as $e^1(0)$, and the vector that have the end in the final point as $e^1(1)$. So far as the cylinder is connected, these two points can be connected by a continuous curve $e^1(t)$, $t \in [0;1]$.

Let us introduce now a vector $e^2(t)$ which continuously depends on t . Let us require that the end of the vector $e^2(t)$ always lies on the same cylinder of the type (6). Furthermore, we require that the scalar product $e^1(t) \cdot e^2(t) = const$ does not depend on t . And what is more, we require that the vectors k , $e^1(t)$ and $e^2(t)$ are linearly independent when $t = 0$ and, therefore, they are linearly independent with any t .

The vector $e^2(t)$ is not defined uniquely, of course. It always can be substituted by $e^2(t) + \alpha(t)\, k$, where $\alpha(t)$ is an arbitrary continuous function. We will not eliminate this non-uniqueness in any way.

Let us use the three vectors k , $e^1(t)$ and $e^2(t)$ as a basis in $M^\perp$. Using the three-dimensional version of the first statement from the section 5 we get a continuous family of linear transformations in $M^\perp$ which preserve the scalar product in $M^\perp$.

Let us prove now that these transformations can be completed as transformations from $\mathfrak{L}_k$. For this let us take an arbitrary vector $n(0)$ that does not belong to the subspace $M^\perp$. The vectors k , $e^1(t)$, $e^2(t)$ and $n(0)$ form a basis in the Minkowski space M . Let us show now that for any t vector $n(t)$ can be chosen so that the following four values do not depend on t :

$$k \cdot n(t) = const \ , \qquad e^1(t) \cdot n(t) = const \ , \qquad e^2(t) \cdot n(t) = const \ , \qquad (n(t))^2 = const \ . \tag{7}$$

The first three conditions with fixed t define in the Minkowski space a line $\widetilde{n}$, which is a point of the factor-plane $\widetilde{N}$ constructed in the section 4. As we have pointed out in the section 5, on this line there is exactly one point satisfying the fourth of the equations (7). This is the point where the desired vector $n(t)$ has its end.

Using now the first statement from the section 5, we see that we have constructed a continuous family of transformations from $\mathfrak{L}_k$ transforming a given point of the cylinder (6) to any other given point. So, we have proved the

L e m m a. *Transformations of the little group $\mathfrak{L}_k$ act on the cylinder* (6) *transitively.*

By the way, we have proved the following

S t a t e m e n t. *A linear transformation of the space $M^\perp$, preserving the scalar product there, can be uniquely completed as a linear transformation of M , preserving the scalar product in M .*

Let us suppose now that the representation $T^\perp$ is decomposable, i. e. the space $M^\perp$ decomposes into a direct sum $M^\perp = M' \oplus M''$ of two reducing subspaces M' and M'' . One of these two subspaces, for example M' , necessarily contain some vector that does not belong to $M^\parallel$. The end of this vector lies on the corresponding cylinder (6). But then, as we have just shown, the whole cylinder must belong to M' . But the linear shell of the cylinder (6) coincides with $M^\perp$ and, therefore, $M' = M^\perp$. So, we have proved a

L e m m a. *Every reducing subspace of the representation $T^\perp$ either coincides with $M^\perp$ or belongs to $M^\parallel$.*

From this we have the

T h e o r e m. *The representation $T^\perp$ is indecomposable.*

Complex vector representation

9. Complexification. As we have shown in the paper [IV], for investigation of electromagnetic field it is useful to study also the complex vector representation of the little group $\mathfrak{L}_k$. In order to get this representation we just should admit that components of vectors under consideration can be complex numbers and suppose that representation of the group $\mathfrak{L}_k$ acts there by the same formulas as in the real case. And the group $\mathfrak{L}_k$ does not change in any way.

The complexified Minkowski space will be denoted as $M_{\mathbb{C}}$. Generally, later on all spaces and representations will be labeled by symbol of the corresponding field of scalars. If two spaces are denoted the same but have different scalar label, then we imply, that the complex space is the complexification of the real. For example, the real Minkowski space will be denoted as $M_{\mathbb{R}}$. We will also imply that real spaces are subsets of corresponding complex spaces. The representation of the group $\mathfrak{L}_k$ that acts in $M_{\mathbb{C}}$ will be denoted as $T_{\mathbb{C}}$.

An arbitrary vector $n \in M_{\mathbb{C}}$ can be represented as a sum of its real and imaginary parts: $n = n^{\mathrm{Re}} + in^{\mathrm{Im}}$. Here $n^{\mathrm{Re}}, n^{\mathrm{Im}} \in M_{\mathbb{R}}$. So far as transformations of the group $\mathfrak{L}_k$ are transformations with real coefficients, we may suppose that a transformation of a complex vector is a simultaneous transformations of its real and imaginary parts.

All reducing subspaces and factor-spaces found in the real case are obviously complexified. But the analogy between the representations $T_{\mathbb{R}}$ and $T_{\mathbb{C}}$ is not full. Reducibility and decomposability of complexified representations must be investigated in addition. For proving of indecomposability of complex representations the following lemma will play an important role

L e m m a. *Let us suppose that we have defined: a vector* $n \in M_{\mathbb{C}}$ *; some two-dimensional subspace* $P^2_{\mathbb{R}}$ *in* $M_{\mathbb{R}}$ *, such that* $n \in P^2_{\mathbb{C}}$ *; and some one-dimensional subspace* $P^1_{\mathbb{R}}$ *in* $P^2_{\mathbb{R}}$ *, such that* $n \notin P^1_{\mathbb{C}}$ *. Then multiplying* n *by an appropriate factor* $\theta \in \mathbb{C}$ *we can make that* $(\theta n)^{\mathrm{Im}} \in P^1_{\mathbb{R}}$ *and* $(\theta n)^{\mathrm{Re}} \notin P^1_{\mathbb{R}}$ *.*

In order to prove this lemma, let us multiply the initial vector n by a number like $e^{i\varphi}$, where the argument φ runs through the set of real numbers. When the argument φ changes, the vectors $n^{\mathrm{Re}}(t)$ and $n^{\mathrm{Im}}(t)$ will move on some ellipsis in the plane $P^2_{\mathbb{R}}$ (If n^{Re} and n^{Im} are parallel, the ellipsis degenerates into the segment). This ellipsis intersects with any one-dimensional subspace $P^1_{\mathbb{R}}$ belonging to $P^2_{\mathbb{R}}$. Now it is obvious that with appropriate φ the imaginary part of the vector will be in $P^1_{\mathbb{R}}$ and the real part will not be there.

10. Decomposability of $T^{\perp/\parallel}_{\mathbb{C}}$. As we have shown in the section 6, in the space $M^{\perp/\parallel}_{\mathbb{R}}$ the little group $\mathfrak{L}_k$ acts as the group $SO(2)$. So far as the group $SO(2)$ is Abelian, as it follows from the Schur lemma, the complex representation $T^{\perp/\parallel}_{\mathbb{C}}$ must be reducible. On the other hand, so far as the group $SO(2)$ is compact, the representation $T^{\perp/\parallel}_{\mathbb{C}}$ is equivalent to some unitary representation and therefore it is fully reducible. The corresponding subrepresentations we will denote as $T^{(+1)/\parallel}_{\mathbb{C}}$ and $T^{(-1)/\parallel}_{\mathbb{C}}$.

In order to make this result more concrete, let us introduce in the real space $M^{\perp/\parallel}_{\mathbb{R}}$ an orthonormal basis $\{\tilde{e}^i\}_{i=1,2}$. Then the matrices of the representation $T^{\perp/\parallel}_{\mathbb{R}}$ will take the form:

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

If now in the complex plane $M^{\perp/\parallel}_{\mathbb{C}}$ we use complex “spiral” basis $\{\tilde{e}^{(\lambda)}\}_{\lambda=\pm1}$:

$$\tilde{e}^{(\pm1)} = \frac{\tilde{e}^1 \pm i\,\tilde{e}^2}{\sqrt{2}} ,$$

the matrices of the representation $T^{\perp/\parallel}_{\mathbb{C}}$ become diagonal:

$$\begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{+i\varphi} \end{pmatrix}$$

So, we have proved the

T h e o r e m. *The representation* $T^{\perp/\parallel}_{\mathbb{C}}$ *has exactly two reducing subspaces:* $M^{(+1)/\parallel}_{\mathbb{C}}$ *and* $M^{(-1)/\parallel}_{\mathbb{C}}$ *. The sum of the corresponding subrepresentations is the representation* $T^{\perp/\parallel}_{\mathbb{C}}$ *:*

$$T^{\perp/\parallel}_{\mathbb{C}} = T^{(+1)/\parallel}_{\mathbb{C}} \oplus T^{(-1)/\parallel}_{\mathbb{C}} .$$

For the electromagnetic field (see [IV]) this theorem means that there exist plane waves with helicity $+1$ and -1 — the well-known fact.

Let us notice also that:

a. So far as in the Minkowski space we have not initially introduced any orientation, there is no orientation on the plane $M_{\mathbb{R}}^{\perp/\|}$. Therefore we can not say which space is denoted $M_{\mathbb{C}}^{(+1)/\|}$, and which $M_{\mathbb{C}}^{(-1)/\|}$. In fact, by fixation of these notations we introduce orientation.

b. If we extend the Lorentz group to the *full* Lorentz group by including there space reflections, then the little group also expands. The group $SO(2)$ acting in the plane $M_{\mathbb{R}}^{\perp/\|}$ expands to $O(2)$. Under reflections the complex spaces $M_{\mathbb{C}}^{(+1)/\|}$ and $M_{\mathbb{C}}^{(-1)/\|}$ will be transformed to each other and the complex representation in the space $M_{\mathbb{C}}^{\perp/\|}$ will become irreducible.

11. Subrepresentations $T_{\mathbb{C}}^{(+1)}$ and $T_{\mathbb{C}}^{(-1)}$. In accordance with the decomposability of the factor-representation $T_{\mathbb{C}}^{\perp/\|}$ shown in the section 10, in the complex Minkowski space $M_{\mathbb{C}}$ we can find two two-dimensional reducing subspaces of the representation $T_{\mathbb{C}}$. These two subspaces we will denote $M_{\mathbb{C}}^{(+1)}$ and $M_{\mathbb{C}}^{(-1)}$. The corresponding one-dimensional factor-spaces $M_{\mathbb{C}}^{(+1)}/M_{\mathbb{C}}^{\|} = M_{\mathbb{C}}^{(+1)/\|}$ and $M_{\mathbb{C}}^{(-1)}/M_{\mathbb{C}}^{\|} = M_{\mathbb{C}}^{(-1)/\|}$ were introduced in the section 10.

Now it is clear that in the complex case the series (1) can be thickened in two ways:

$$\{0\} = M_{\mathbb{C}}^{0} \subset M_{\mathbb{C}}^{\|} \subset M_{\mathbb{C}}^{(+1)} \subset M_{\mathbb{C}}^{\perp} \subset M_{\mathbb{C}}^{4} = M_{\mathbb{C}} \ , \tag{8}$$

$$\{0\} = M_{\mathbb{C}}^{0} \subset M_{\mathbb{C}}^{\|} \subset M_{\mathbb{C}}^{(-1)} \subset M_{\mathbb{C}}^{\perp} \subset M_{\mathbb{C}}^{4} = M_{\mathbb{C}} \ , \tag{9}$$

It will be shown in the section 13, that the complex vector representation does not have other reducing subspaces. So, besides (8) and (9), the representation $T_{\mathbb{C}}$ does not have other composition series.

12. Indecomposability of representation $T_{\mathbb{C}}$. Let us suppose that the representation $T_{\mathbb{C}}$ is decomposable, i. e. the space $M_{\mathbb{C}}$ can be represented as a sum

$$M_{\mathbb{C}} = M'_{\mathbb{C}} \oplus M''_{\mathbb{C}} \tag{10}$$

of two reducing subspaces $M'_{\mathbb{C}}$ and $M''_{\mathbb{C}}$. Then one of these subspaces, for example $M'_{\mathbb{C}}$, necessarily contains some vector n that does not belong to $M_{\mathbb{C}}^{\perp}$. According to the lemma from the section 9, without loss of generality we can assume that $n^{\mathrm{Im}} \in M_{\mathbb{R}}^{\perp}$ and $n^{\mathrm{Re}} \notin M_{\mathbb{R}}^{\perp}$.

Let us introduce now some real vector e^1 . In the case if the vector n^{Im} does not belong to $M_{\mathbb{R}}^{\|}$, e^1 is just another notation for n^{Im} . But if n^{Im} belongs to $M_{\mathbb{R}}^{\|}$, vector e^1 is chosen as any vector such that $e^1 \in M_{\mathbb{R}}^{\perp}$ and $e^1 \notin M_{\mathbb{R}}^{\|}$.

Consider now the subgroup of the little group $\mathcal{L}_k$ that leaves the vector e^1 unchanged. According to the definition of the vector e^1 , this subgroup in any case leaves unchanged the vector n^{Im} . From the results of the section 8 we get that the elements of this group can be defined by some vector e^2 , which always has its end lying on some cylinder (6), the condition $e^1 \cdot e^2 = const$ is satisfied, and the vectors k , e^1 and e^2 are linearly independent. The end of the vector e^2 runs through some line parallel to k : $e^2(t) = e^2(0) + tk$. The equations (7) for the vector $n^{\mathrm{Re}}(t)$ take the form:

$$k \cdot n^{\mathrm{Re}}(t) = const \ , \qquad e^1 \cdot n^{\mathrm{Re}}(t) = const \ , \qquad e^2(t) \cdot n^{\mathrm{Re}}(t) = const \ , \qquad (n^{\mathrm{Re}}(t))^2 = const \ .$$

The third equation can be written with more details:

$$(e^2(0) + tk) \cdot n^{\mathrm{Re}}(t) = const \ . \tag{11}$$

Obviously, if we suppose that $(n^{\mathrm{Re}}(t) - n^{\mathrm{Re}}(0)) \in M^{\|}$, then the equation (11) for $t \neq 0$ can not be satisfied. Therefore, the subspace $M'_{\mathbb{C}}$ must contain some real vector belonging to $M_{\mathbb{R}}^{\perp}$ and not belonging to $M_{\mathbb{R}}^{\|}$. But then, according to the second lemma in the section 8, we have $M_{\mathbb{R}}^{\perp} \subset M'_{\mathbb{C}}$ and, therefore, $M_{\mathbb{C}}^{\perp} \subset M'_{\mathbb{C}}$.

Let us come back again to the complex vector $n \in M'_{\mathbb{C}}$, for which $n^{\mathrm{Re}} \notin M_{\mathbb{R}}^{\perp}$ and $n^{\mathrm{Im}} \in M_{\mathbb{R}}^{\perp}$. So far as we have already proved that $M_{\mathbb{C}}^{\perp} \subset M'_{\mathbb{C}}$, we get that in $M'_{\mathbb{C}}$ there is also such a vector which has zero imaginary part and which has the real part equal to the real part of vector n . In other words, we have proved that $M'_{\mathbb{C}}$ contains a real vector that does not belong to $M^{\perp}_{\mathbb{R}}$. But then, according to the lemma of the section 7, $M_{\mathbb{R}} \subset M'_{\mathbb{C}}$, and, therefore, spaces $M'_{\mathbb{C}}$ and $M_{\mathbb{C}}$ coincide.

So, we have proved the

L e m m a. *Every reducing subspace of the representation $T_{\mathbb{C}}$ either coincides with the whole space $M_{\mathbb{C}}$ or belongs to $M^{\perp}_{\mathbb{C}}$.*

From this we get the

T h e o r e m. *The representation $T_{\mathbb{C}}$ is indecomposable.*

13. Indecomposability of representations $T^{\perp}_{\mathbb{C}}$, $T^{(+1)}_{\mathbb{C}}$ and $T^{(-1)}_{\mathbb{C}}$. Let us prove the following lemma:

L e m m a. *Every subrepresentation of representation $T^{\perp}_{\mathbb{C}}$ contains the subrepresentation $T^{\parallel}_{\mathbb{C}}$.*

In order to prove this lemma, let us denote the subrepresentation under consideration as $T'_{\mathbb{C}}$. The reducing subspace where it acts we denote $M'_{\mathbb{C}}$.

Consider some non-zero vector n from the subspace $M'_{\mathbb{C}}$. If it lies in $M^{\parallel}_{\mathbb{C}}$, then the statement of the lemma is satisfied. Suppose now that this vector does not belong to $M^{\parallel}_{\mathbb{C}}$.

If the vectors k , n^{Re} and n^{Im} are linearly dependent then, according to the lemma from the section 9, we can without loss of generality think that $n^{\mathrm{Im}} \in M^{\parallel}_{\mathbb{R}}$ and $n^{\mathrm{Re}} \notin M^{\parallel}_{\mathbb{R}}$. According to the first lemma from the section 8, using transformations of the little group $\mathfrak{L}_k$ we can change the vector n^{Re} so that we add to it an arbitrary vector from $M^{\parallel}_{\mathbb{R}}$. At the same time the vector n^{Im} remains unchanged. Therefore $M^{\parallel}_{\mathbb{R}} \subset M'_{\mathbb{C}}$ and, therefore, $M^{\parallel}_{\mathbb{C}} \subset M'_{\mathbb{C}}$.

But if the vectors k , n^{Re} and n^{Im} are linearly dependent, then, according to the statement of the section 8, choosing a proper transformation from the little group $\mathfrak{L}_k$ we can add to the vector n^{Re} a vector parallel to k so that the vector n^{Im} remains unchanged. Therefore, again $M^{\parallel}_{\mathbb{R}} \subset M'_{\mathbb{C}}$ and $M^{\parallel}_{\mathbb{C}} \subset M'_{\mathbb{C}}$. So, the lemma is proved.

From this lemma we get the two theorems:

T h e o r e m. *The representation $T^{\perp}_{\mathbb{C}}$ is indecomposable.*

T h e o r e m. *The representations $T^{(+1)}_{\mathbb{C}}$ and $T^{(-1)}_{\mathbb{C}}$ are indecomposable.*

Furthermore, from the lemma and the theorem of the section 6 we get the

T h e o r e m. *Besides $M^{\parallel}_{\mathbb{C}}$, $M^{(+1)}_{\mathbb{C}}$, $M^{(-1)}_{\mathbb{C}}$ (and $M^{\perp}_{\mathbb{C}}$) the representation $T^{\perp}_{\mathbb{C}}$ does not have any other reducing subspaces.*

So, the representation $T_{\mathbb{C}}$ does not have any other composition series, besides (8) and (9).

14. Matrices of representations $T_{\mathbb{R}}$ and $T_{\mathbb{C}}$. Many of the obtained results can be represented by patterns for matrices of the representations $T_{\mathbb{R}}$ and $T_{\mathbb{C}}$. If in the spaces $M_{\mathbb{R}}$ and $M_{\mathbb{C}}$ we choose appropriate bases, then these templates become block-triangular:

$$\begin{pmatrix} \boxed{1} & \cdot & \cdot & \cdot \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{\boxed{SO(2)}}} & \cdot \\ 0 & & & \cdot \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix} \qquad \begin{pmatrix} \boxed{1} & \cdot & \cdot & \cdot \\ 0 & \boxed{e^{-i\varphi}} & 0 & \cdot \\ 0 & 0 & \boxed{e^{+i\varphi}} & \cdot \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

And it is known that on the place of every point we have non-zero element, in general case.

Studying in what subspaces and factor-spaces different blocks of these matrices act I leave for the reader.

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