

THE UNCERTAINTY RELATION BETWEEN ENERGY AND TIME IN NON-RELATIVISTIC QUANTUM MECHANICS

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(Received February 22, 1945)

A uncertainty relation between energy and time having a simple physical meaning is rigorously deduced from the principles of quantum mechanics. Some examples of its application are discussed.

1. Along with the uncertainty relation between coordinate q and momentum p one considers in quantum mechanics also the uncertainty relation between energy and time.

The former relation in the form of the inequality

$$\Delta q \cdot \Delta p \geq \frac{h}{2}, \quad (1)$$

where Δq and Δp are respective standards** and h —Planck's constant divided by 2π , follows, as well known, directly from the quantum mechanical formalism. As regards the usual considerations referring to the so-called Heisenberg's microscope, to the determination of velocity by means of the Doppler effect, etc., their aims consist essentially but in the elucidation of the connection between the measurements of coordinates and momenta and the formalism of the quantum mechanics.

An entirely different situation is met with in the case of the relation

$$\Delta H \cdot \Delta T \sim h, \quad (2)$$

where ΔH is the standard of energy, ΔT —a certain time interval, and the sign \sim denotes that the left-hand side is at least of the order of the right-hand one.

In order to establish this relation, one usually refers, on one hand, to the relation energy = $h\nu$, and, on the other hand, to the trivial relation $\Delta\nu \cdot \Delta T \sim 1$, connecting the "uncertainty" $\Delta\nu$ in the measurement of the frequency of a monochromatic vibration with the time interval ΔT , during which this measurement is carried out.

It has, however, more than once been pointed out, that in non-relativistic quantum mechanics it is consistent to consider the energy as an "observable" in Dirac's sense, corresponding to the Hamiltonian of the given mechanical system. If one accepts this definition of energy, one cannot, of course, identify energy with the frequency of a monochromatic vibration multiplied by h . Therefore, the above derivation of the relation (2)

* The manuscript of this paper was almost completely prepared for publication when Prof. Mandelstam suddenly died on November the 27th, 1944.

** Standard denotes the square root of the average quadratic deviation from the mean value.

becomes invalid and this relation itself becomes meaningless.

Apparently no general derivation of the relation (2) (or of any other similar uncertainty relation referring to energy) from the fundamental principles of quantum mechanics has yet been given. Instead certain special cases were considered, the quantities ΔH and ΔT were defined in some special way, varying from case to case, and it was shown, that the quantities thus defined satisfy the relation (2).

2. The object of the present note is to indicate a very general uncertainty relation for the energy, which, just as (1), follows from the quantum formalism if one takes into account also the Schrodinger equation.

It is based on the following consideration. It is known that the total energy of an isolated quantum mechanical system, in distinction to a classical one, does not, in general, have a definite and constant value. Instead of this the *probability* to obtain in a measurement any specified value of the energy of the system remains constant in time. This is just the content of the law of conservation of energy in quantum mechanics.

Energy is exactly determined only in the special case of a stationary state. But in this case, as easily seen, *all* dynamical variables or, more exactly, their distribution functions, remain constant in time. In other words, the *definiteness* of the total energy of the system entails the *constancy* with respect to the time of all dynamical variables. It can be concluded that there must exist a general connection between the dispersion of the total energy of the system and the time variation of coordinates, momenta, etc. The uncertainty relation with which we are concerned gives a quantitative formulation of this connection.

Let R and S denote any two quantities and at same time the corresponding Hermitian operators. Following two relations are known to hold:

$$\Delta S \cdot \Delta R \geq \frac{1}{2} |\overline{RS - SR}|, \quad (3)$$

where ΔS and ΔR are the standards of the quantities S and R and the horizontal bar denotes as usual the quantum mechanical average, and

$$h \frac{\partial \bar{R}}{\partial t} = i \overline{(HR - RH)}, \quad (4)$$

where H is the Hamiltonian of the system not depending explicitly on the time.

Putting in (3) $S \equiv H$ we obtain with the help of (4) the desired uncertainty relation for the energy in the form of the following inequality:

$$\Delta H \cdot \Delta R \geq \frac{h}{2} \left| \frac{\partial \bar{R}}{\partial t} \right|. \quad (5)$$

This relation gives, thus, the connection between the standard ΔH of the total energy of an isolated system, the standard ΔR of some other dynamical quantity and the rate of change of the average value of this quantity.

The relation (5) can be put in a somewhat different form. The absolute value of an integral cannot exceed the integral of the absolute value of the integrand. Hence integrating (5) from t to $t + \Delta t$ and taking into account that ΔH is constant one gets

$$\Delta H \cdot \Delta t \geq \frac{h}{2} \frac{|\overline{R_{t+\Delta t}} - \overline{R_t}|}{\Delta R}, \quad (5a)$$

where the denominator of the right-hand side denotes the average value of the standard ΔR during the time Δt .

Sometimes (especially in the case of a continuous spectrum of eigenvalues) it is convenient to refer the variations of the average value of a dynamical quantity to its standard. This enables one in a number of cases to estimate the effectiveness of these variations. This can be illustrated by a somewhat similar situation which is met with when one estimates the resolving power of optical instruments.

In such cases it is convenient to introduce a special notation — ΔT — for the shortest time, during which the average value of a certain quantity is changed by an amount equal to the standard of this quantity. ΔT can be called the standard time.

With the help of this notation one can rewrite equation (5a) in the following form:

$$\Delta H \cdot \Delta T \geq \frac{h}{2}. \quad (5b)$$

It follows from (5a) that for the variation of a quantity it is necessary not only that $\Delta H \neq 0$, but that the average standard of this quantity should also be different from

zero (if* $\Delta H \neq \infty$). Thus a dynamical quantity cannot vary if its dispersion remains equal to zero; this result is obvious in the case of a discrete spectrum but not in the case of a continuous one. It follows further from (5a) that if at a certain instant the dispersion of a quantity R vanishes, whereas its average value does not remain constant, then initially, *i. e.* for small Δt , the standard ΔR varies much more rapidly than R .

The characteristic feature of the uncertainty relation (5) for the energy consists in the fact that it contains an arbitrary quantity R , its physical meaning depending thus on the choice of this quantity. The neglect of this circumstance is apparently responsible for the fact, that different problems to which the relation (2) has been applied remained in a number of cases without any mutual connection.

3. In order to illustrate the applications of the relation (5), we shall consider three examples.

Let us first consider an one-dimensional wave packet. If we put $R \equiv q$, then \bar{R} is the centre of gravity of the packet, while ΔR can be considered as its average width and ΔT as the time of its passage. The relation (5b) shows, that the precision of the localization in time of the passage of the packet through a point of space depends on the dispersion of the total energy of the system and cannot be high if this dispersion is small, the quantitative relation between ΔH and ΔT being given by (5b).

This example is usually considered in connection with the relation (2), the result stated being arrived at under the assumption that the motion takes place in absence of external forces. Equation (5b) shows, that this result holds for the motion in an arbitrary potential field of force.

The second and the third examples, which we are going to consider, have also been often discussed in connection with the relation (2).

This discussion was, however, based on new definitions of ΔH and ΔT , so that it has had no connection whatever with the first example. In our treatment all the three examples are particular cases of the relation (5).

Let us consider a system of two degrees of freedom and let its Hamiltonian be of the type:

$$H(q_1, q_2) = H_1(q_1) + H_2(q_2) + \mu H_{12}(q_1, q_2).$$

If μ is small or if H_{12} is of the type of the interaction energy of two elastically colliding particles, then one usually speaks of two systems, each having one degree of freedom, which are in interaction with one another.

Under these assumptions $\bar{H}_1 + H_2 \cong \bar{H}$; under certain conditions (resonance) \bar{H}_1 and \bar{H}_2 may vary considerably with time, whereas their sum remains approximately constant. In this case the energy is said to pass from one system to another, \bar{H}_1 being called the energy of the first and \bar{H}_2 — the energy of the second system.

Let us put $R \equiv H_2$; then (5a) shows that, in general, the transition of the energy takes place the slower the smaller the dispersion of the total energy, or more exactly: the standard time of the transition of the energy from one system to the other is not smaller than $h/2\Delta H$.

This case can be illustrated by the one-dimensional collision of two elastic particles of a vanishing size and of equal mass, one of which has initially a velocity differing from zero, with a small dispersion, while the other is nearly at rest.

The collision process can be approximately described in the plane q_1 and q_2 (q_1 and q_2 are the coordinates of the particles) by the reflection of a set of waves, limited with respect both to its length and width, from a mirror placed at an angle of 45° to the direction of q_1 , the "wave line" of the set being normal to q_1 and the length and width of the set being very large compared with the wave length. The length of the set of waves will be assumed to be much greater than its width. Before the collision the wave function consists of a wave set which is propagated along q_1 towards the mirror. Until the set has reached the mirror $H_2 \sim 0$. Then the front part of the set is turned by reflection over an angle of 90° . So long as the time which has elapsed from the beginning of the reflect-

* Not infrequently one has to consider states, the energy standard ΔH of which is infinite (for example when the energy is distributed according to the dispersion formula $\frac{\text{const}}{(H-H_0)^2 + \Gamma^2}$). Since the relation (5) becomes in such cases meaningless, it would be desirable to find a more general relation of the same type as (5).

ion is small, the reflected portion of the set contains a small number of waves. In other words, in agreement with the general deduction from equation (5a) mentioned above, the rate of change of ΔH_2 at this stage is greater than that of H_2 , so that ΔH_2 is relatively large, while \bar{H}_2 is still small.

With the further propagation of the wave set the ratio $\bar{H}_2/\Delta H_2$ increases until the whole set is turned in the direction q_2 . It can also easily be seen that with the decrease of the dispersion of the energy of the whole system, which corresponds, for example, to an increase of the length of the initial wave set, all the corresponding time intervals (in particular the standard time) are increased, which is also in qualitative agreement with (5b).

An entirely different meaning has the well known relation which follows from the perturbation theory:

$$|(H_1 + H_2) - (H_{10} + H_{20})| \sim \frac{\hbar}{t}, \quad (6)$$

where H_{10} and H_{20} denote the initial energies of the interacting systems (or particles) 1 and 2 at the instant $t = 0$, while H_1 and H_2 are their energies at an instant t .

The quantity $H' = H_1 + H_2$, which will be called the proper energy of the particles, is by no means equal to the total energy of the system $H = H_1 + H_2 + \mu H_{12}$, so that the relation (6) has nothing to do with the uncertainty ΔH of the total energy of the system, which for any isolated system remains constant in time.

Moreover, even if one considers not the total energy H , but only the proper energy of the particles H' the relation (6), contrary to the widespread opinion, does not mean that the uncertainty of the proper energy of the particles decreases with the increase of the time t , during which these particles interact with each other. In fact, the probability, that under the influence of a perturbation μH_{12} a transition of the system from the initial state ψ_0 with a proper energy $H'_0 = H_{10} + H_{20}$ into a state with a proper energy $H' = H_1 + H_2$ will take place during a time t , is proportional to the oscillating function of time

$$\sin^2[(H' - H'_0)t/2\hbar]/(H' - H'_0)^2.$$

Hence the probability $\omega(\epsilon, t)$ that H' will at the instant t differ from H'_0 by an amount

not smaller than a fixed quantity ϵ does not tend to zero as t increases.

The problem can, however, be stated in a different way. Let us divide the results of the measurements of the state of the system at the instant t into two classes—class *A*, comprising the cases when the system at the instant t was found in the initial state ψ_0 , and class *B*, comprising all other cases. As t increases, the probability of the cases *A* decreases, while the transitions of the system into states, which satisfy the law of conservation of the proper energy H' , become the more prevailing the greater t (resonance). This is just why the probability $\omega(\epsilon, t)$ defined above does not substantially vary with the time, in spite of the decrease of the cases *A*, corresponding to an exact conservation of the proper energy of the particles.

In other words, if the cases of class *A* are set aside and the relative probability of different results of the measurements within the class *B* (comprising the states of the system, which are different from the initial one) are considered, then this relative probability $\omega_B(\epsilon, t)$ of the results of the measurements of the class *B*, for which the quantity $(H'_0 - H'_0)$ at the instant t is not smaller than a fixed quantity ϵ , will decrease with the time. Formula (6) establishes the connection between t and that value $\epsilon = |(H_1 + H_2) - (H_{10} + H_{20})|$, for which the probability $\omega_B(\epsilon, t)$ thus defined becomes comparable with unity.

We have not succeeded thus far to establish a connection between this relation (6) and the uncertainty relation (5)–(5b), although we are inclined to believe that such a connection does exist.

As a third example we shall consider the width of spectral lines or, more generally, the relation between the lifetime of a given state ψ_0 of a system and the uncertainty ΔH of the energy of this state.

Let L denote the projection operator corresponding to the state ψ_0 of the system, defined by the relation

$$L\psi = (\psi_0\psi) \cdot \psi_0, \quad \text{where } (\psi_0\psi) = \int \psi_0^* \psi dx.$$

One of the eigenvalues of the operator L is equal to unity, while all others are equal to zero. Hence

$$L^2 = L. \quad (7)$$

The average value \bar{L} is equal to the probability that the system is in the state ψ_0 ; evidently $\bar{L} \leq 1$. According to (7) $\Delta L = \sqrt{\bar{L}^2 - (\bar{L})^2} = \sqrt{\bar{L} - (\bar{L})^2}$. Hence the relation (5) assumes for the operator L the following form*:

$$\Delta H \cdot \sqrt{\bar{L} - (\bar{L})^2} \geq \frac{\hbar}{2} \left| \frac{\partial \bar{L}}{\partial t} \right|. \quad (8)$$

This inequality contains only one variable quantity $\bar{L}(t)$ and its derivative and can easily be integrated.

If for instance $\bar{L}(0) = 1$ (i. e. if at the instant $t=0$ the system was certain to be in the state ψ_0), then it follows from (8) that for $t \geq 0$

$$\frac{\pi}{2} - \arcsin \sqrt{\bar{L}(t)} \leq \frac{\Delta H \cdot t}{\hbar}. \quad (9)$$

Hence, for $0 \leq t \leq \pi \hbar / 2 \Delta H$

$$\bar{L}(t) \geq \cos^2 \left(\frac{\Delta H \cdot t}{\hbar} \right) \quad (10)$$

[if $t > \pi \hbar / 2 \Delta H$, then (9) does not restrict the value of $\bar{L}(t)$, since at any rate $0 \leq \bar{L}(t) \leq 1$]. If τ denotes the half life of the state ψ_0 [i. e. $\bar{L}(\tau) = 1/2$ if $\bar{L}(0) = 1$], then (10) yields the relation

$$\tau \cdot \Delta H \geq \frac{\pi}{4} \hbar, \quad (11)$$

between τ and the uncertainty ΔH of the energy of the state ψ_0 , which is somewhat more precise than the usual one.

It should be mentioned that in a number of problems referring to *measurements*, the relation (5) often enables one to estimate the time interval which under given conditions is necessary to attain a sufficient "precision" of the measurements.

4. The uncertainty relation (1) is usually derived for the so-called "pure case", i. e. on the assumption, that the state of the system

can be described by a wave function. This assumption underlies also the derivation of the relation (5) given above.

However, both (1) and (5) hold also for the general case of a "mixture". This can easily be proved as follows.

It is known that the general case of a mixture

$$\bar{R} = \sum p_i \int \varphi_i^*(R\varphi_i) dv, \quad (12)$$

where φ_i are certain functions which can be considered as wave functions of the pure cases composing the mixture, while p_i are positive numbers satisfying the condition $\sum p_i = 1$.

We shall use a horizontal bar without an index to denote the average value of a quantity F for the whole mixture. Let further $\bar{F}^i = \int \varphi_i^*(F\varphi_i) dv$. Then according to (12)

$$\begin{aligned} \bar{R} &= \sum p_i \bar{R}^i \quad \text{and} \quad (\Delta R)^2 = \overline{(R - \bar{R})^2} = \\ &= \sum p_i \overline{(R - \bar{R}^i)^2}. \end{aligned} \quad (13)$$

We shall also use the notation $(\Delta_i R)^2 = \overline{(R - \bar{R}^i)^2}$. According to (13)

$$(\Delta R)^2 = \sum p_i \{ \bar{R}^{2i} - 2\bar{R} \cdot \bar{R}^i + (\bar{R})^2 \}.$$

Now the i th term of the sum is not smaller than * $p_i \overline{(R - \bar{R}^i)^2} = p_i (\Delta_i R)^2$, so that

$$(\Delta R)^2 \geq \sum p_i (\Delta_i R)^2. \quad (14)$$

Let R and S denote any two quantities and let

$$(\Delta_i R)^2 (\Delta_i S)^2 \geq a_i^2, \quad (15)$$

then the inequality holds

$$(\Delta R)^2 (\Delta S)^2 \geq \left(\sum p_i a_i \right)^2. \quad (16)$$

* It follows namely from $(R^i - \bar{R})^2 \geq 0$ that $-2\bar{R}\bar{R}^i + (\bar{R})^2 \geq (\bar{R}^i)^2$.

* Since according to (8) and in agreement with a well known result of the perturbation theory $\partial \bar{L} / \partial t = 0$ if $\bar{L} = 1$ (or $\bar{L} = 0$), the exponential law of decay of a state $\bar{L} = e^{-\gamma t}$ cannot hold at small t 's; according to (8) this law can set in only when t exceeds the value $t_0 = \frac{1}{\gamma} \ln \left(1 + \frac{\hbar^2 \gamma^2}{4 \Delta H^2} \right)$.

In fact, according to (14) and (15)

$$\begin{aligned} (\Delta R)^2 (\Delta S)^2 &\geq \sum p_i (\Delta_i R)^2 \cdot \sum p_k (\Delta_k S)^2 \geq \\ &\geq \sum_{i,k < i} p_i p_k \left\{ \left(\frac{\Delta_i R}{\Delta_k R} \right)^2 a_k^2 + \left(\frac{\Delta_k R}{\Delta_i R} \right)^2 a_i^2 \right\} + \\ &\quad + \sum_i p_i^2 a_i^2. \end{aligned}$$

Noting that the term of the first sum of the right-hand side with the indices i and k is not smaller than $2p_i p_k a_i a_k$, one immediately obtains (16).

Let $S \equiv q$, $R \equiv p$. According to (1) and (15) $a_i = \hbar/2$. It follows from (16) that the relation (1) remains valid for a mixture, since ΔR and ΔS are the standards of R and S referred to the whole mixture.

In order to apply (16) to a generalization of the relation (5) one must take into account that all functions φ_i are solutions of the same Schrodinger equation. Letting $S \equiv H$ and comparing (5) with (15) one can put

$$a_i^2 = \frac{\hbar^2}{4} \left(\frac{\partial \bar{R}^i}{\partial t} \right)^2. \quad (17)$$

It can easily be shown that

$$\partial \bar{R} / \partial t = \sum p_i \partial \bar{R}^i / \partial t.$$

Hence, taking into account (16) and (17), we see that the uncertainty relation (5) is valid in the general case of a mixture.

Translated by S. Frenkel.